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Articles

Part I.– Curved Earth Geometrical Optics

Part II.– The Concept of the Equivalent Radius of the Earth in Tropospheric Propagation

Part III.– Ground-Wave Propagation Over an Inhomogeneous Smooth Earth. Part I. Theoretical Considerations

Part IV.– Ground-Wave Propagation Over an Inhomogeneous Smooth Earth. Part 2. Experimental Evidence and Practical Implications

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The ideal one strives for can best be measured by the sacrifices it requires. To be honest, my ideal is the creator of the universe. I am captivated by it and have never really concerned myself with non-ideals. Why should I? I don't understand it at all, and it makes no sense to me. These non-ideals would rather destroy me and ruin me mentally. And finally, they make their problems my problems, problems that I was never aware of and never came into contact with. Empirically confirmed, this is the only thing they are capable of and an integral part of their personality. Especially Germans always need a scapegoat to blame for their problems. And they generally do so with malicious and deeply insidious actions. This is the unpleasant truth! Empirically validated with my experience in the 21st century: Ultimately, Germans are still largely the same as they were back then. Nothing in principle has changed. The plebs or proletariat! Also the bourgeoisie! Today, as then, almost all of them are forged in the same furnace. Just on another level. Bear this in mind! This is NO joke! Yesterday it was me, tomorrow it's you! Just because you do not fit well in cracker-barrel rallying cries, beer is not your favorite beverage and you are not interested in big cars and sports, yodeling beer tent parties in Lederhosen, where "traditional" songs are sung and roast pork with dumplings. And if you have harmed somebody or not doesn't matter.

Jan

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Part I.

First Paper

Curved Earth Geometrical Optics

Abstract

In this paper the effect of the earth's curvature in the geometric-optical treatment of propagation within the visual range is presented as a correction to the flat earth geometry. Factors to he applied to the flat earth values of the angle of elevation at the point of reflection and of the path difference between the direct and reflected waves are derived by a simple graphical process, and also the divergence factor arising from the increased divergence on reflection from the convex surface of the earth is obtained. Asymptotic values are given for points very near to the horizon.

1. Introduction

In the theory of wave propagation over a flat earth, the signal strength at a distance from a transmitter can be represented as the sum of a space wave determined by the method of geometrical optics and a surface wave of the SOMMERFELD type. On long waves, the latter predominates even at considerable heights, hut on ultra-short waves it decreases rapidly with height, and is only of significance when both transmitter and receiver are very close to the ground. For most practical purposes, the space wave alone need be considered in Very High Frequency (VHF) propagation, and by the method of geometrical-optics, it can be represented as the sum of a direct free space wave and one reflected from the earth's surface.

When the curvature of the earth is taken into account, the field-strength nearly up to the horizon can still he determined by the method of geometrical optics, though in the neighborhood of the horizon there must be a transition from the geometric-optical solution to the full wave value of the diffracted field below the line of sight. We shall confine ourselves here to the region where the geometric-optical solution is valid, merely indicating the limit of its applicability as a function of the wave-length.

The curvature of the earth, however, does modify the geometry of the problem, and introduces a divergence factor because a cone of rays incident on a convex surface is more divergent after reflection than before. It is the purpose of this paper to show that these effects of the earth's curvature can be presented as simple corrections to the flat earth values, obtainable by a simple graphical process. We shall not consider in detail the actual value of the field-strength obtained by adding together the direct and reflected waves.

Usually we can assume that the difference in path length is too small to affect the relative amplitudes of the waves, and that the angle of elevation at the reflection point is small enough to ensure that the field vectors are effectively parallel ; so that in adding the waves vectorially, one has only to allow for the decrease in amplitude at reflection and that due to the divergence factor, and for the change in phase on reflection and that introduced by the path difference between the direct and reflected waves. In any case, the curvature corrections cease to be significant when these conditions do not hold, and, when the heights of the transmitter and receiver are comparable with the distance between them, and the angle of elevation is no longer small, the problem of compounding the field-strengths of the direct and reflected waves can be treated on its own merits, as a flat earth problem.

2. Flat Earth Results

As we propose to present the curvature effect as a correction to the flat earth results, a brief derivation of the latter follows. The geometry is shown in Figure 2.1, which serves to define the notation used.

The angle of elevation θ is given by

$$\tan(\theta) = \frac{h_1}{d_1} = \frac{h_2}{d_2} = \frac{h_1 + h_2}{d} \tag{2.1}$$

and the position of the point of reflection C is fixed by the condition that the total distance d is divided in the ratio of h_1 to h_2.

For the path difference $p = r_1 + r_2 - R$ we have

$$\begin{aligned} R^2 &= r_1^2 + r_2^2 + 2r_1r_2 \cdot \cos(2\theta) \\ &= (r_1 + r_2)^2 - 4r_1r_2 \cdot \sin^2(\theta) \end{aligned}$$

so that

$$p = \frac{4r_1r_2 \cdot \sin^2(\theta)}{r_1 + r_2 + R}\,.$$

When θ is small, we may replace r_1 and r_2 in this formula by d_1 and d_2 and R by d, so that

$$p = \frac{2d_1d_2}{d} \cdot \theta^2 \tag{2.2}$$

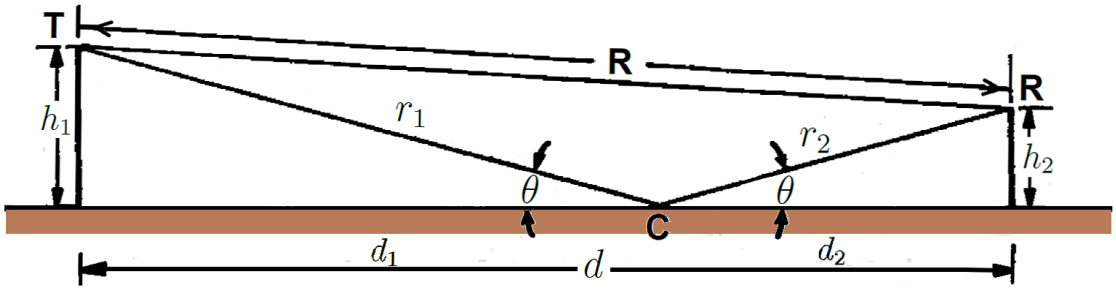

Figure 2.1.: Illustration of derivation of the correction accounting for the flat earth results

and when we use Equation 2.1 for a small θ, we have as the flat earth value for p

$$p = \frac{2h_1h_2}{d}\,. \tag{2.3}$$

In terms of phase angle χ for a wave-length λ, this gives

$$\chi = \frac{2\pi \cdot p}{\lambda} = \frac{4\pi \cdot h_1h_2}{\lambda \cdot d}$$

and the field-strength $\mathfrak{E}$ in terms of the free space value $\mathfrak{E}_0$ for the direct wave will be given by

$$\mathfrak{E} = \mathfrak{E}_0 + \rho \cdot \mathfrak{E}_0 \cdot e^{-\mathrm{i}\,\chi}\,,$$

where we have assumed a time factor $\exp(\mathrm{i}\,\omega t)$, and ρ is the reflection coefficient which is a function of the earth constants and the angle θ, but is nearly equal to -1 (i.e., no reduction in amplitude, but a change in phase of π) for VHF propagation.

For the curved earth, this formula is modified to

$$\mathfrak{E} = \mathfrak{E}_0 + \rho \cdot \mathcal{D}\,\mathfrak{E}_0 \cdot e^{-\mathrm{i}\,\chi}\,,$$

where $\chi = \frac{2\pi \cdot p}{\lambda}$, but p has a modified value, and $\mathcal{D}$ is the divergence factor. Theoretically the reflection coefficient, which is obtained by satisfying the boundary conditions at the surface of the earth, depends on the curvature of the earth, but the effect is negligible at high frequencies.

3. Curved Earth Geometry

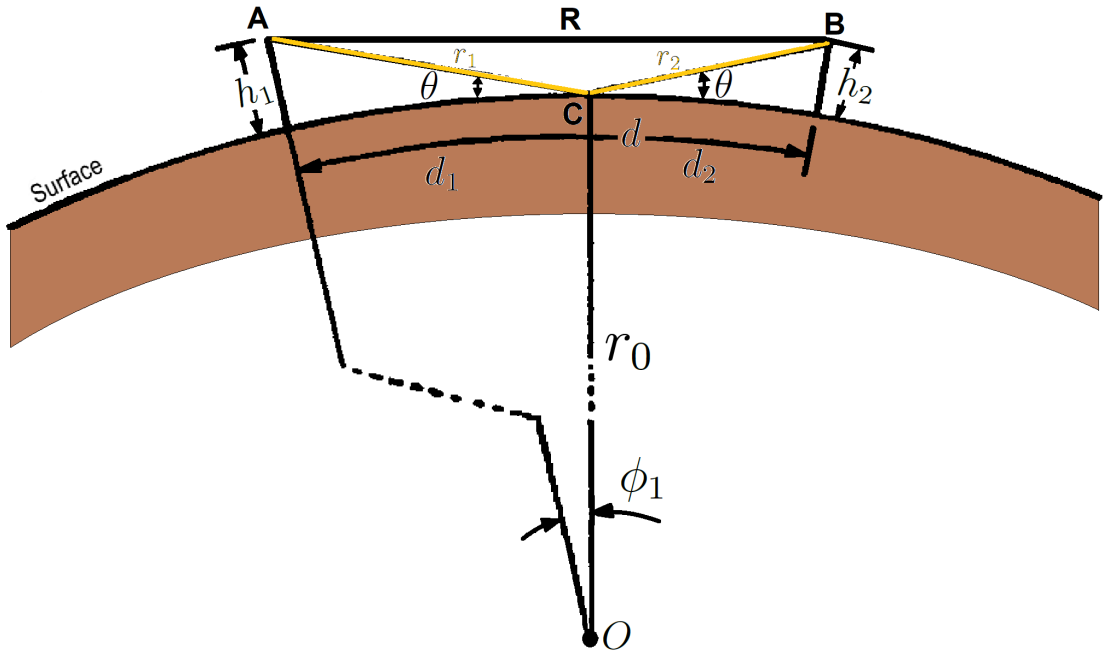

Figure 3.1.: Illustration of derivation taking curvature of the earth into account

When we allow for the curvature of the earth, Figure 2.1 is replaced by Figure 3.1. As the relations developed above are reciprocal with respect to h_1 and h_2, and the reciprocity condition applies equally to the curved earth case, we shall not stipulate which is the transmitter and which the receiver, but shall designate the terminals by A and B. Now from $\triangle ACO$, if r_0 is the radius of the earth,

$$\frac{r_0}{r_0 + h_1} = \frac{\cos(\theta + \phi_1)}{\cos(\theta)} = \cos(\phi_1) - \tan(\theta) \cdot \sin(\phi_1)$$

so that

$$\tan(\theta) \cdot \sin(\phi_1) = \frac{h_1}{r_0 + h_1} - 2 \cdot \sin^2\left(\frac{\phi_1}{2}\right)$$

and therefore

$$\tan(\theta) = \frac{h_1}{r_0 + h_1} \cdot \csc(\phi_1) - \tan\left(\frac{\phi_1}{2}\right), \tag{3.1}$$

where

$$\phi_1 = \frac{d_1}{r_0}\,.$$

As in our case $h_1 \ll r_0$ and $\phi_1 \ll 1$, we may write

$$\tan(\theta) = \frac{h_1}{d_1} - \frac{d_1}{2 \cdot r_0}\,. \tag{3.2}$$

The accurate form in Equation 3.1 is useful in the analogous problem of reflections from the ionosphere, where we may not have $h_1 \ll r_0$. The relation for $\tan(\theta)$ in Equation 3.2 leads at once to the usual approximation for the horizon distance by putting $\theta = 0$, i.e., $d_H^2 = 2 \cdot r_0 h_1$ and for our own case of raised transmitter and receiver, the total horizon distance becomes

$$d_H = \sqrt{2 \cdot r_0 h_1} + \sqrt{2 \cdot r_0 h_2} = \sqrt{2 \cdot r_0} \cdot \left[\sqrt{h_1} + \sqrt{h_2}\right]\,. \tag{3.3}$$

From the equality of the angles of incidence and reflection at C we have

$$\tan(\theta) = \frac{h_1}{d_1} - \frac{d_1}{2 \cdot r_0} = \frac{h_2}{d_2} = \frac{d_2}{2 \cdot r_0}\,. \tag{3.4}$$

This is an equation for determining d_1 and d_2, remembering that $d_1 + d_2 = d$, the given distance between the transmitter and receiver. It is now convenient to choose h_1 and h_2 so that $h_1 > h_2$, and it follows that $d_1 > d_2$, though we no longer have the simple relation in Equation 2.1. We may now define a positive quantity, α, by:–

$$\alpha = \frac{h_1 - h_2}{h_1 + h_2}\,, \tag{3.5}$$

which ranges from 0 to 1 as $\frac{h_2}{h_1}$ ranges from 1 to 0. In terms of α we may write

$$h_1 = \frac{h_1 + h_2}{2} \cdot (1 + \alpha) \tag{3.6}$$

and

$$h_2 = \frac{h_1 + h_2}{2} \cdot (1 - \alpha)\,. \tag{3.7}$$

Analogously, we may write

$$d_1 = \frac{d}{2} \cdot (1 + \beta) \tag{3.8}$$

and

$$d_2 = \frac{d}{2} \cdot (1 - \beta)\,, \tag{3.9}$$

where β is a positive quantity to be determined from the solution of Equation 3.4, using these values. Like α, it ranges from 0 to 1, as $\frac{h_2}{h_1}$ ranges from 1 to 0. If we

substitute these values of h_1, d_1, h_2 d_2, into Equation 3.4, we obtain

$$\frac{h_1+h_2}{d}\cdot\left(\frac{1+\alpha}{1+\beta}\right)-\frac{d\cdot(1+\beta)}{4\cdot r_0}=\frac{h_1+h_2}{d}\cdot\left(\frac{1-\alpha}{1-\beta}\right)-\frac{d\cdot(1-\beta)}{4\cdot r_0}$$

so that

$$\frac{2\cdot(h_1+h_2)\cdot(\alpha-\beta)}{d\cdot(1-\beta^2)}=\frac{2\cdot\beta\cdot d}{4\cdot r_0}\,.$$

Thus

$$\alpha=\beta+k\cdot\beta\cdot\left(1-\beta^2\right)\,, \tag{3.10}$$

where

$$k\equiv\frac{d^2}{4r_0\cdot(h_1+h_2)}\,. \tag{3.11}$$

This quantity k is known in terms of the given d, h_1 and h_2, and as d must be less than the horizon value d_H in Equation 3.3, it follows that k is always less than unity except on the horizon when $h_1 = h_2$.

Equation 3.10 is a cubic for determining β in terms of the known α and k, but we can obtain a very simple and useful graphical solution by regarding it as an equation giving α as a function of k for various values of β, as we shall obtain a series of straight lines. When $\beta = 0$, we shall have the line $\alpha = 0$, and when $\beta = 1$, the line $\alpha = 1$; agreeing with our limiting values. Intermediately, the slope of the lines increases to a maximum of 0.385 when $\beta = \frac{1}{\sqrt{3}}$ and then decreases again. When $k = 0$, $\beta = \alpha$, so that the lines cut the ordinate at α-values equal to their corresponding β-values, and the α scale serves to mark the β-values for the various lines. This construction limits the solution of the cubic equation to the one root relevant to our problem. The system of lines at intervals of 0.02 for β is shown in Figure 3.2.

Now, if we have a given pair of α and k values, we define a point on the appropriate β-line, and if we move along this line towards larger values of α and k, we must eventually arrive at an (α, k) point, which would correspond to a set of values for which d would be the horizon distance, d_H, for the heights, h_1 and h_2. Let us suppose the value of k at which this occurs on our β-line is k_0. Then, for different values of β, these k_0 end points of the β-lines will define a boundary curve, as in Figure 3.2, which marks the edge of the visual range, and any set of d, h_1 and h_2-values giving a (α, k) point lying outside this boundary, corresponds to a system in which the receiver is below the line of sight of the transmitter.

To find the value of k_0 in terms of α, so that we can construct the boundary curve, it is simplest to return to the value of $\tan(\theta)$ in Equation 3.4 which we can

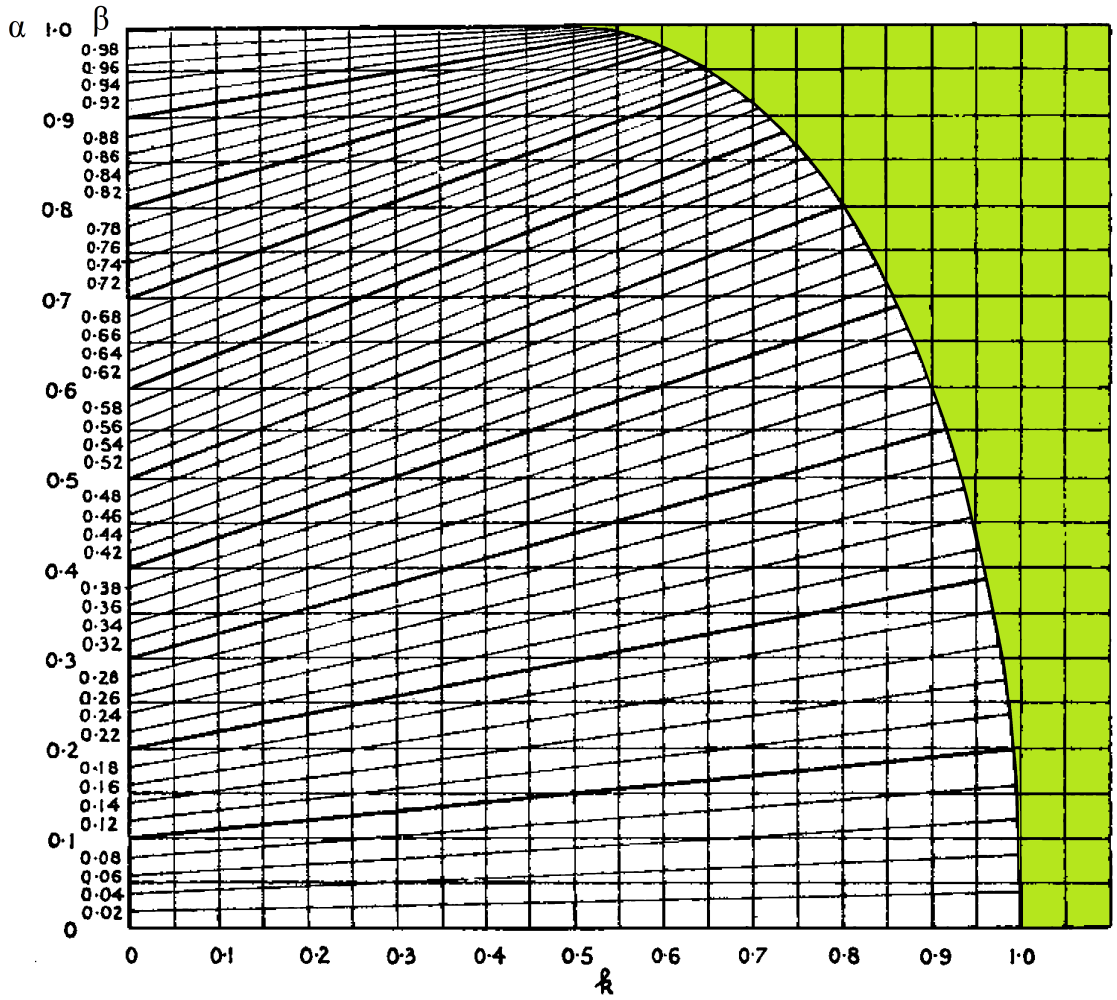

Figure 3.2.: Graphical solution of the cubic relation Equation 3.10 with α as a function of k defined according to Equation 3.11 for various values of β.

▭ ≡ receiver is below the line of sight of transmitter

write as

$$\tan(\theta) = \frac{h_1 + h_2}{d} \cdot \left[\frac{(1+\alpha)}{(1+\beta)} - k \cdot (1+\beta) \right] .$$

We may express θ in terms of k and β by substituting for α from Equation 3.10, giving

$$\tan(\theta) = \frac{h_1 + h_2}{d} \cdot \left[1 - k \cdot \left(1 + \beta^2 \right) \right] . \tag{3.12}$$

For our given value of β, k_0 is then that value of k, that makes $\theta = 0$, i.e.

$$k_0 \equiv \frac{1}{(1+\beta^2)} . \tag{3.13}$$

Thus, as we should expect, k_0 is simply a function of β, and ranges from 1 to 0.5 as β ranges from 0 to 1. We can now find the relation between α and k_0 by replacing k by k_0 and β by $\sqrt{\frac{1-k_0}{k_0}}$ in Equation 3.10 giving

$$\alpha = 2 \cdot \sqrt{k_0 \cdot (1 - k_0)} ,$$

from which the boundary curve can be obtained.

We may regard k_0 simply as a function of β given by Equation 3.13, which as a matter of convenience we may find graphically by running along the β-line to the boundary curve and reading the value on the k-scale in Figure 3.2. This graph allows us to transfer our attention from α to β, and hence to k_0, so that instead of defining our system by α and k, we can work in terms of k_0 and k. If we return to Equation 3.12 and use Equation 3.13 we have

$$\tan(\theta) = \frac{h_1 + h_2}{d} \cdot \gamma , \tag{3.14}$$

where

$$\gamma = 1 - \frac{k}{k_0} , \tag{3.15}$$

and by comparison with Equation 2.1, we see that Equation 3.14 gives $\tan(\theta)$ as the flat earth value modified by a factor γ, which from Equation 3.15 is a function of k and k_0. If we plot γ as a function of k for various values of k_0, we get a series of straight lines shown in Figure 3.3. We notice that the lines cut the abscissa scale where $k = k_0$, so that the values of k_0 can be read directly from the k-scale, showing the advantage of working in terms of k_0 rather than of β. They are drawn for every 0.02 of k_0 from 0.5 to 1, and, like the lines in Figure 3.2, are very suitable for interpolation.

Coming now to the value of the path difference, p, the analysis is formally the same as before as far as Equation 2.2, but we must now use the value of θ, assumed

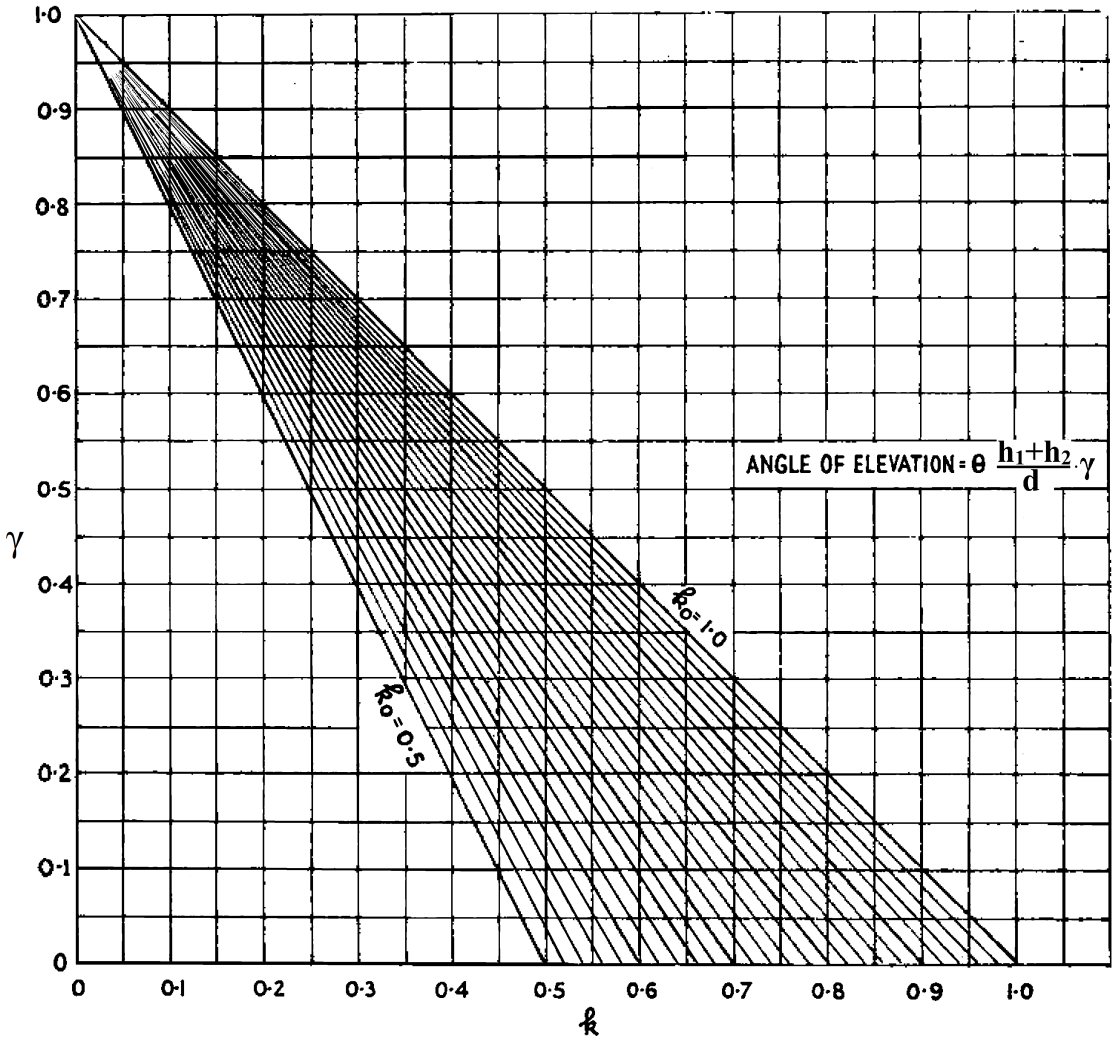

Figure 3.3.: Plot of γ, given by Equation 3.15 as a function of k, defined according to Equation 3.11, for various values of k_0 as defined by Equation 3.13

small, in Equation 3.14, giving

$$p = \frac{2 \cdot d_1 d_2}{d} \cdot \left(\frac{h_1 + h_2}{d} \right)^2 \cdot \gamma^2 \, .$$

Bearing in mind the flat earth value for p in Equation 2.3, we write from Equation 3.6 and Equation 3.7:–

$$(h_1 + h_2)^2 = \frac{4 \cdot h_1 h_2}{(1 - \alpha^2)}$$

and from Equation 3.8 and Equation 3.9:–

$$d_1 \cdot d_2 = \frac{d^2}{4} \cdot \left(1 - \beta^2\right) \, ,$$

so that

$$p = \frac{2 \cdot h_1 h_2}{d} \cdot \eta \, , \tag{3.16}$$

where

$$\eta \equiv \frac{(1 - \beta^2)}{(1 - \alpha^2)} \cdot \gamma^2 \, . \tag{3.17}$$

Thus p is given by the flat earth value modified by a factor η.
From Equation 3.10, we find

$$\left(1 - \alpha^2\right) = \left(1 - \beta^2\right) \cdot \left[1 - 2k \cdot \beta^2 - k^2 \cdot \beta^2 \cdot \left(1 - \beta^2\right)\right]$$

and using Equation 3.13 to express β^2 in terms of k_0, we find

$$\frac{(1 - \alpha^2)}{(1 - \beta^2)} = \frac{(k_0 - k)^2 + k_0 k \cdot [2 \cdot (k_0 - k) + k \cdot (2k_0 - 1)]}{k_0^2} \, .$$

Finally, by substituting for γ from Equation 3.15, we have from Equation 3.17:

$$\eta = \frac{(k_0 - k)^2}{(k_0 - k)^2 + k_0 k \cdot [2 \cdot (k_0 - k) + k \cdot (2k_0 - 1)]} \, , \tag{3.18}$$

which is a form suitable for computation, since $(k_0 - k)$ and $(2k_0 - 1)$ are both positive. When $k_0 = 0.5$, then $\eta = (1 - 2 \cdot k)$; and when $k_0 = 1$, then $\eta = (1 - k)^2$. The curves for η as a function of k are shown in Figure 3.4 at intervals of 0.05 for k_0. Here again $k_0 = k$ on the abscissa scale, and the curves are suitable for interpolation.

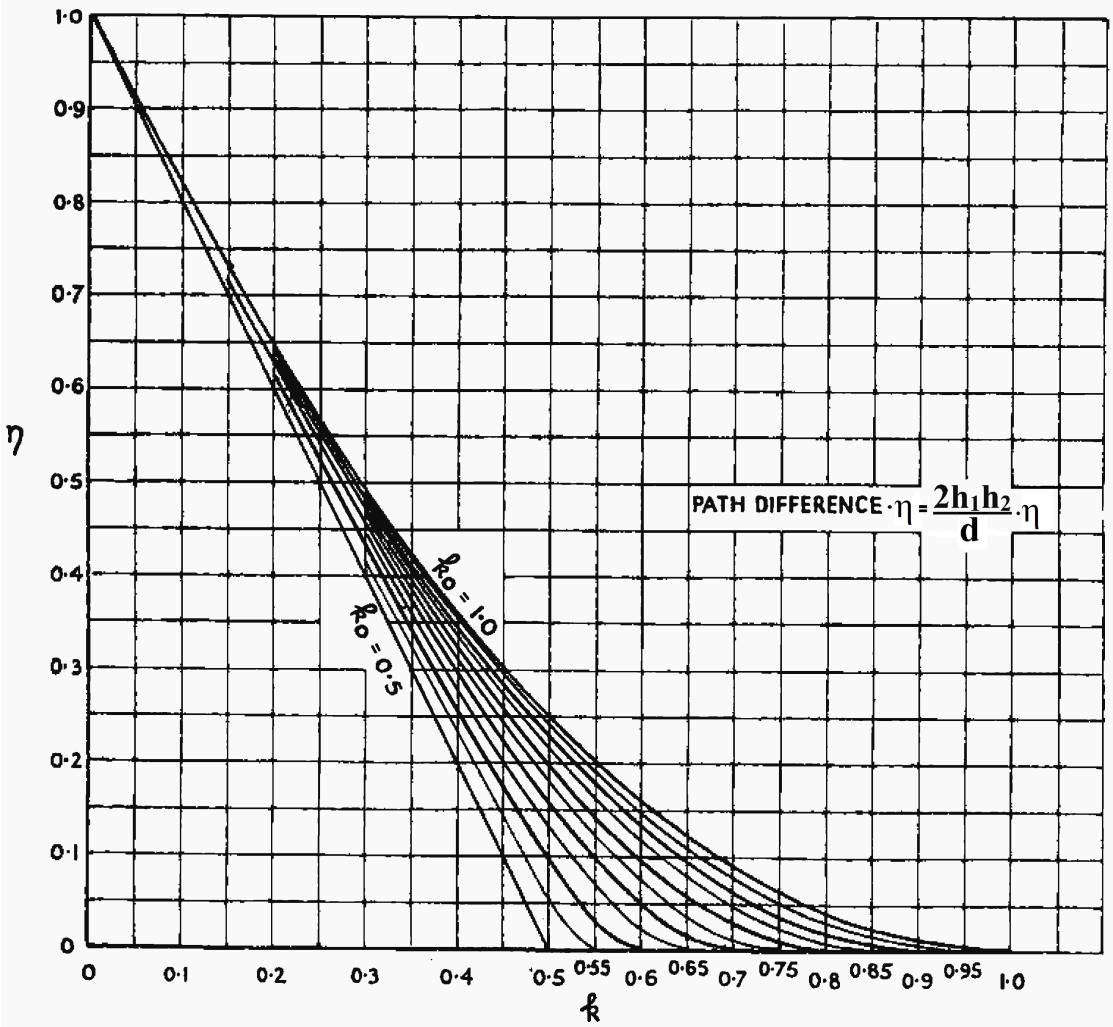

Figure 3.4.: Curves for η given by Equation 3.18 as a function of k, defined according to Equation 3.11

4. The Divergence Factor

We shall find that the divergence factor is a symmetrical function of h_1 and h_2 and can be expressed in terms of k_0 and k, but in deriving it we must consider a cone of rays diverging from the transmitter, which for convenience we will take at A in Figure 3.1. We now have Figure 4.1 in which we consider a ray AC' at an angle $\delta\psi$ to AC, incident at a point C' distant δd beyond C. At C the cone has a width $CD = r_1 \cdot \delta\psi$, which after reflection at a flat earth would become b, given by

$$b = (r_1 + r_2) \cdot \delta\psi \tag{4.1}$$

at the receiver B.

After reflection from the curved earth, the width of the cone at B will be some larger quantity, b', and as the reflected ray remains in the same vertical plane as the incident ray, there will be no increase in width in the azimuthal direction, so that the area at B is increased in the ratio of b' to b. As the divergence factor $\mathcal{D}$ is referred to the wave amplitude and not to the intensity, it will be given by

$$\mathcal{D} = \sqrt{\frac{b}{b'}}\,. \tag{4.2}$$

Now, after reflection, the rays will diverge at some new angle $\delta\psi'$ greater than $\delta\psi$, and the beam width $C'D' = CD$ will be increased at B by an amount $r_2 \cdot \delta\psi'$ so that we have

$$b' = r_1 \cdot \delta\psi + r_2 \cdot \delta\psi'\,. \tag{4.3}$$

As the tangent plane at C' is rotated from that at C by the angle $\delta\phi$ subtended by δd at the centre of the earth, the divergence of the reflected ray is increased by $2 \cdot \delta\phi$, so that

$$\delta\psi' = \delta\psi + 2 \cdot \delta\phi\,. \tag{4.4}$$

From Equation 4.3 and Equation 4.4 we now have

$$b' = (r_1 + r_2) \cdot \delta\psi + 2r_2 \cdot \delta\phi\,.$$

Finally, to relate $\delta\phi$ to $\delta\psi$, we have from Figure 4.1

$$\delta d = r_0 \cdot \delta\phi = r_1 \cdot \delta\psi \cdot \csc(\theta)\,,$$

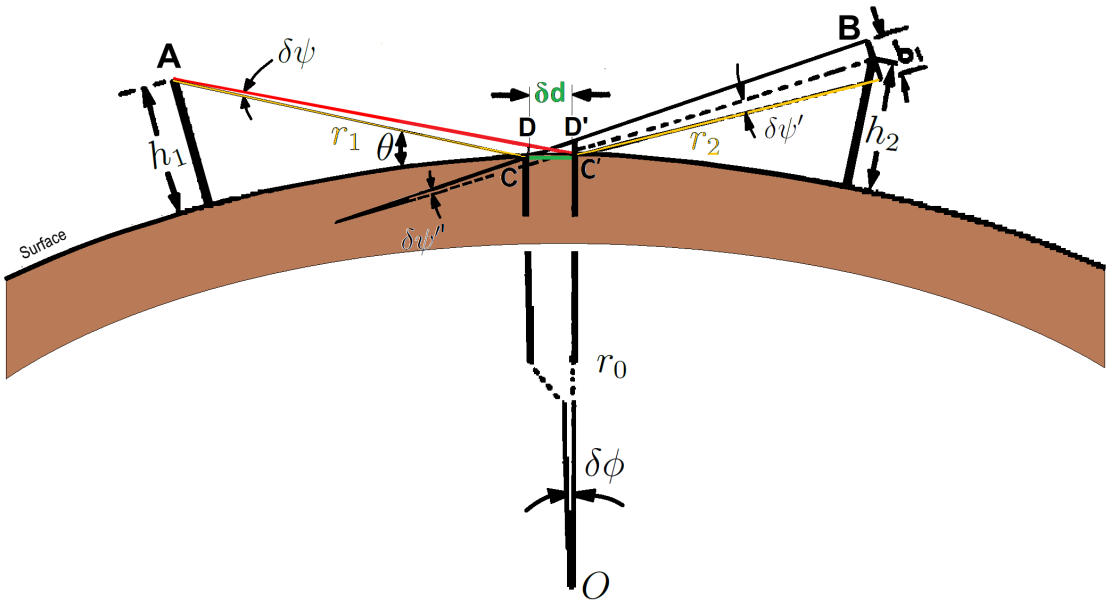

Figure 4.1.: Consideration of a ray AC' (red line) over a curved earth with radius r_0

i.e.,

$$\delta\phi = \frac{r_1}{r_0} \cdot \delta\psi \cdot \csc(\theta)\,.$$

Thus

$$b' = (r_1 + r_2) \cdot \delta\psi + \frac{2 \cdot r_1 r_2}{r_0} \cdot \csc(\theta) \cdot \delta\psi\,,$$

and from Equation 4.1 and Equation 4.2 we now have

$$\mathcal{D} = \sqrt{\frac{r_1 + r_2}{r_1 + r_2 + \frac{2 \cdot r_1 r_2}{r_0} \cdot \csc(\theta)}}\,,$$

which is a symmetrical function of r_1 and r_2 as anticipated. As before, we replace r_1 and r_2 by d_1 and d_2 and take θ small, and obtain

$$\mathcal{D} = \frac{1}{\sqrt{1 + \frac{2 \cdot d_1 d_2}{r_0 \cdot d \cdot \theta}}}\,. \tag{4.5}$$

From Equation 3.8, Equation 3.9 and Equation 3.14 we may write

$$\frac{2 \cdot d_1 d_2}{r_0 \cdot d \cdot \theta} = \frac{2 d_2 \cdot (1 - \beta^2)}{4 r_0 \cdot (h_1 + h_2)} \cdot \gamma = \frac{2k \cdot (1 - \beta^2)}{\gamma}\,,$$

and when we express β^2 in terms of k_0 and give γ its value in Equation 3.15 we

have on substitution in Equation 4.5:

$$\mathcal{D} = \sqrt{\frac{(k_0 - k)}{(k_0 - k) + 2k \cdot (2k_0 - 1)}}\,. \tag{4.6}$$

When we plot $\mathcal{D}$ as a function of k for various values of k_0, we get the curves shown in Figure 4.2, and we see that they have some rather striking features. In particular, they all go through the same point on the ordinate $k = 0.5$; for on putting $k = 0.5$, we find that $\mathcal{D} = \frac{1}{\sqrt{3}}$, irrespective of the value of k_0. When $k_0 = 1$, then

$$\mathcal{D} = \sqrt{\frac{(1 - k)}{(1 + k)}}$$

and when $k_0 = 0.5$, $\mathcal{D} = 1$ for all values of k up to $k = k_0$ on the horizon. If we take a value of k_0 very slightly greater than 0.5, $\mathcal{D}$ is very nearly unity for values of k less than 0.5, but then drops of rapidly through the value of $\frac{1}{\sqrt{3}}$ at $k = 0.5$ to zero at $k = k_0$.

The discontinuity which occurs when $k_0 = 0.5$ has a simple physical significance. When both h_1 and h_2 are not zero, $\mathcal{D} \to 0$ at the limit of the visual range, where $k = k_0$; but if $h_2 = 0$, so that $\alpha = \beta = 1$, and $k = 0.5$, then physically $\mathcal{D}$ remains unity however close we get to the limit of the visual range. If we raise the transmitter or receiver a very small but finite amount above the ground, we do not appreciably increase the visual range, but as h_2 is no longer zero, $\mathcal{D}$ must become zero at the limit of the visual range.

It will be seen that $\mathcal{D}$ cannot be read very accurately when it is small, for values of $k > 0.5$, because of the steepness of the curves, but physically we do not need to determine $\mathcal{D}$ very accurately when it is small, because it then weighs little in determining the total field at the receiver, since it makes that contribution of the reflected wave small compared with that of the direct wave. We do, however need to know $\mathcal{D}$ accurately when it is nearly equal to unity. In the special case when h_2 is very nearly to zero, so that β is nearly unity, $\mathcal{D}$ may remain close to unity over most of the range $k = 0$ to $k = 0.5$, and the curves in Figure 4.2 may not be sufficiently accurate or suitable for interpolation. It is not a long process to compute any special case from Equation 4.6. More usually $\mathcal{D}$ is very nearly unity through k being small, rather than through β being nearly unity, and in this case, $\mathcal{D}$ is approximately given by

$$\mathcal{D} = 1 - k \cdot \left[\frac{2k_0 - 1}{k_0}\right]$$

which gives a direct measure of the significant difference $(1 - \mathcal{D})$.

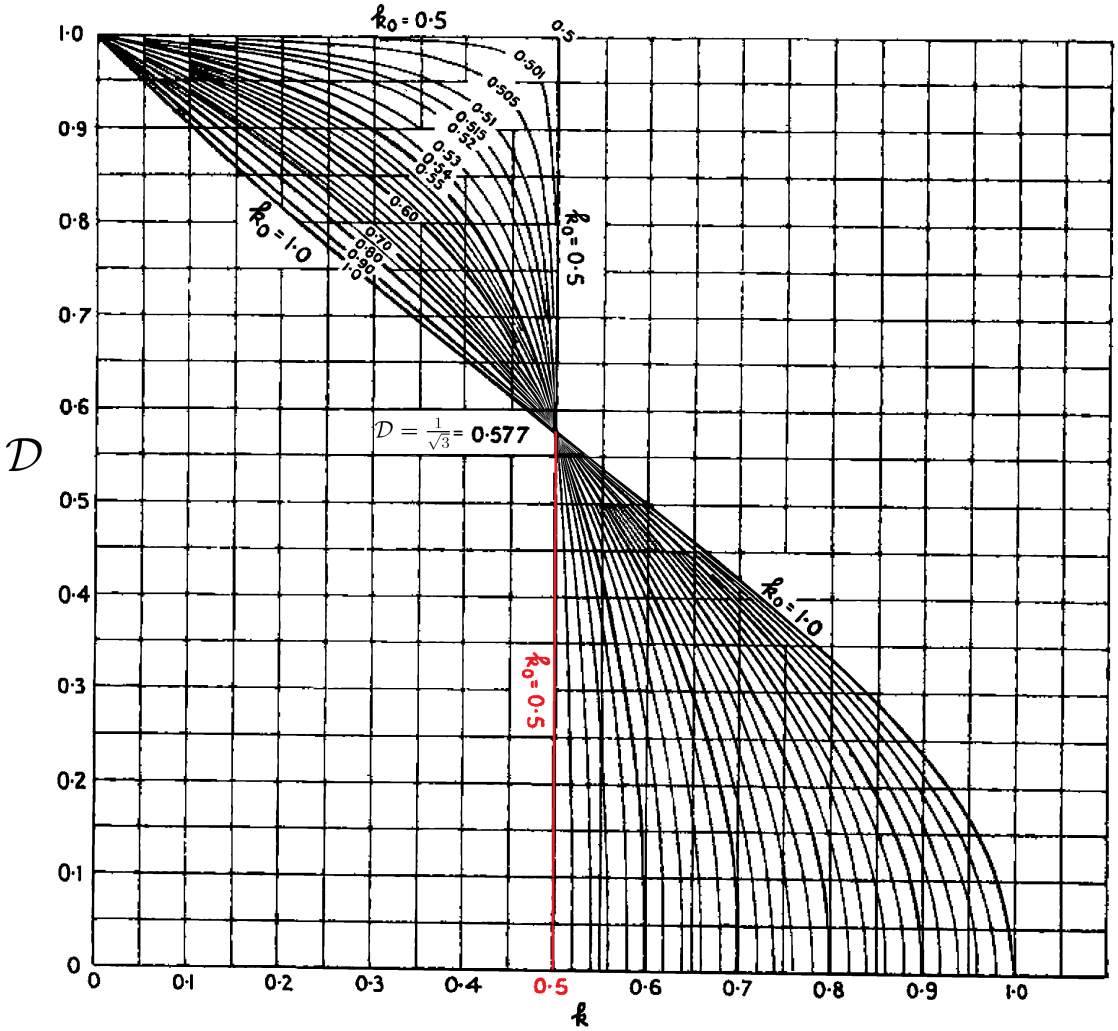

Figure 4.2.: Plot of the divergence factor $\mathcal{D}$ calculated with Equation 4.6 as a function of k as defined in Equation 3.11 for various values of k_0, given by Equation 3.13

5. Values close to the Horizon

In studying the transition from the geometric-optical field-strength to the diffraction values it may be necessary to work with values of d nearly equal to the horizon distance d_H, and although under these conditions the path difference p may be very small, it may still give rise to an appreciable phase angle $\chi = \frac{2\pi \cdot p}{\lambda}$, if λ is very small. It is then better to compute the angle θ and the path difference p directly as functions of the small distance $(d_H - d)$, rather than by Equation 3.14 and Equation 3.16, where the correction factors γ and η are very small quantities and cannot be read accurately from Figure 3.3 and Figure 3.4. In effect we treat the horizon as the datum, and measure distances from it, both in studying the geometric-optical field within the horizon, and the diffraction field beyond.

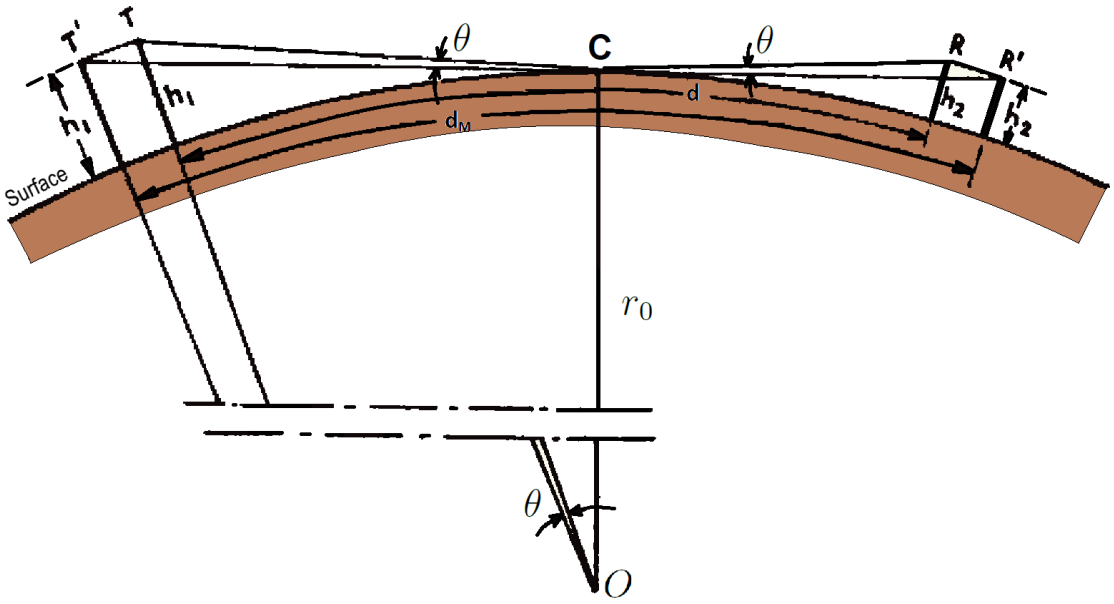

Figure 5.1.: Illustration to determine the value of the angle of elevation θ

The value of θ can be determined very simply from the geometry of Figure 5.1. Here we adopt the artifice of regarding the point of reflection as fixed, and then moving the transmitter and receiver out until they are on the line of sight passing through this point of reflection. If we consider that the transmitter is moved from T to T', remaining at the same height, it follows that the circle on $T'O$ as the diameter passes through C, since $T'CO$ is a right angle, and also through T', since

$\angle T'TO$ is effectively a right angle, when TT' is very small, as in the case we are considering. Thus $\angle TCT'$, i.e. θ, is equal to the angle subtended by $T'T$ at O, and we may write $T'T$ as $r_0 \cdot \theta$. By a similar argument we have for the receiver $RR' = r_0 \cdot \theta$, and so $TT' + RR' = 2r_0 \cdot \theta$. But this must equal the distance by which the receiver at R is within the visual range of the transmitter at T, i.e. $(d_H - d)$, so that we have

$$\theta = \frac{d_H - d}{2 \cdot r_0} \,. \tag{5.1}$$

For the path difference p we return to the form in Equation 2.2 and use the value for θ in Equation 5.1 giving

$$p = \frac{2 \cdot d_1 d_2}{d} \cdot \left(\frac{d_H - d}{2 \cdot r_0} \right)^2 \,.$$

As we are assuming that $(d_H - d) \ll d_H$, we may write d_1 as $\sqrt{2 \cdot r_0 h_1}$, and d_2 as $\sqrt{2 \cdot r_0 h_2}$, and d in the denominator as d_H, so that

$$p = \frac{\sqrt{h_1 \cdot h_2}}{r_0 \cdot d_H} \cdot (d_H - d)^2 \,. \tag{5.2}$$

In using these limiting values of θ and p, we must take into consideration the validity of the geometric-optical treatment in the neighborhood of the horizon. A study of this transition region shows that the criterion for the use of the geometric-optical method is that the angle of elevation θ at the point of reflection must be greater than a limiting angle θ_0 given by

$$\theta_0 = \sqrt[3]{\frac{\lambda}{2\pi \cdot r_0}} \,, \tag{5.3}$$

where λ is the wave-length. If in a given case this angle corresponds to a value of $(d_H - d)$, which is $\ll d_H$, we may combine Equation 5.1 and Equation 5.3 to give the condition that the geometric-optical method is valid so long as

$$(d_H - d) > \sqrt[3]{\frac{4}{\pi} \cdot r_0^2 \cdot \lambda} \,. \tag{5.4}$$

Thus the formulæ in Equation 5.1 and Equation 5.2 can only be used in practice provided that $d_H \gg \sqrt[3]{\frac{4}{\pi} \cdot r_0^2 \cdot \lambda}$, and if this condition is not satisfied, the graphical solution for θ and p involving γ and η will give the requisite accuracy over the range of validity of the geometric-optical method.

6. Practical Examples

In applying our results to practical cases, it is convenient to adopt the convention that the heights h_1 and h_2 are measured in metres, and the distances and the radius of the earth in kilometres. It is useful to take an effective radius of the earth somewhat greater than the true radius, to allow for the refraction in an atmosphere with a linear reduction of refractive index with height. Here we shall allow for this possibility by replacing r_0 in our equations by $\mathfrak{m} \cdot r_0$ where r_0 is the actual radius, taken as 6370 km, and $\mathfrak{m}$ is a factor which may, for example, be chosen as $\frac{4}{3}$. For numerical purposes our Equation 3.11, Equation 3.14, Equation 3.16 and Equation 3.3 may now be written in the form:

$$k = \frac{d^2}{25.5\,\mathrm{m} \cdot (h_1 + h_2)} \tag{6.1}$$

$$\theta = \left[\frac{h_1 + h_2}{d}\right] \cdot \gamma \cdot 10^{-3} \text{ radians} \tag{6.2}$$

$$p = \left[\frac{2h_1 h_2}{d}\right] \cdot \eta \cdot 10^{-3} \text{ metres} \tag{6.3}$$

$$d_H = 3.57 \cdot \sqrt{\mathfrak{m}} \cdot \left(\sqrt{h_1} + \sqrt{h_2}\right) \text{ kilometres} \tag{6.4}$$

If $(d_H - d) \ll d_H$ and the geometric-optical treatment is still valid, we may use Equation 5.1 and Equation 5.2 which will give θ in radians and p in metres.

As a numerical example we will take $h_1 = 800\,\mathrm{m}$, $h_2 = 200\,\mathrm{m}$, $d = 87.5\,\mathrm{km}$, and $\mathfrak{m} = 1$. From Equation 3.5 we have $\alpha = 0.6$, while from Equation 6.1 we have $k = 0.300$, and from Figure 3.2, we find for $\alpha = 0.6$ and $k = 0.3$ that $\beta = 0.49$. By following along this value of β, we find on the boundary $k_0 = 0.81$. From Figure 3.3, Figure 3.4, and Figure 4.2, we find for $k = 0.3$ and $k_0 = 0.81$ that $\gamma = 0.63$, $\eta = 0.47$, and $\mathcal{D} = 0.76$.

Thus, from Equation 6.2, it follows that

$$\theta = \frac{1000}{87.5} \cdot 0.63 \cdot 10^{-3} = 0.00720 \text{ radians} = 0.413 \text{ degrees}\,,$$

and from Equation 6.3

$$p = \frac{2 \cdot 800 \cdot 200}{87.5} \cdot 0.47 \cdot 10^{-3} = 1.72 \text{ metres}\,.$$

If we wish to find the point of reflection, we have from Equation 3.8

$$d_1 = \frac{87.5}{2} \cdot 1.49 = 65.2\,\text{km}\,,$$

and

$$d_2 = d - d_1 = 87.5 - 65.2 = 22.3\,\text{km}\,.$$

We see that whereas $\frac{h_1}{h_2} = 4$; $\frac{d_1}{d_2} = 2.92$, showing the considerable modification produced by the curvature to the flat earth relation $\frac{h_1}{h_2} = \frac{d_1}{d_2}$.

The condition in Equation 5.4 with our conventions becomes

$$(d_H - d) \geqslant 37.2\ \text{m}^{\frac{2}{3}} \cdot \lambda^{\frac{1}{3}}\ \text{km}\,, \tag{6.5}$$

where λ is measured in metres. The critical angle θ_0 in Equation 5.3 becomes

$$\theta_0 = 0.00292\ \text{m}^{-\frac{1}{3}} \cdot \lambda^{\frac{1}{3}}\ \text{radians}\,. \tag{6.6}$$

If $\lambda = 1\,\text{cm}$, i.e. $0.01\,\text{m}$, the condition in Equation 6.5 implies that, for $\mathfrak{m} = 1$, $(d_H - d) > 8.02$ kilometres. In the example considered above, the horizon distance is given from Equation 6.4 by $151.47\,\text{km}$, which is fairly large compared with $8.02\,\text{km}$, and justifies the use of Equation 5.1 and Equation 5.2 when $(d_H - d)$ is near to the limiting value. As a check, we will take the same heights $h_1 = 800\,\text{m}$ and $h_2 = 200\,\text{m}$ and $(d_H - d) = 8.02\,\text{km}$, so that $d = 151.47 - 8.02 = 143.45\,\text{km}$. We again have $\alpha = 0.6$, and now from Equation 6.1 we find $k = 0.807$. Now from Figure 3.2 we have $\beta = 0.35$ and $k_0 = 0.890$, whence from Figure 3.3, Figure 3.4, and Figure 4.2, $\gamma = 0.095$, $\eta = 0.013$, and $\mathcal{D} = 0.26$. Thus from Equation 6.2

$$\theta = \frac{1000}{143.45} \cdot 0.095 \cdot 10^{-3} = 0.000662 \text{ radians}\,,$$

which compares with the value $\theta = 0.000630$ derived from Equation 6.6. From Equation 6.3 we get

$$p = \frac{2 \cdot 800 \cdot 200}{143.45} \cdot 0.013 \cdot 10^{-3} = 0.029\,\text{m}\,,$$

while from Equation 5.2 we have

$$p = \frac{\sqrt{800 \cdot 200}}{6370 \cdot 151.47} \cdot (8.02)^2 = 0.026\,67\,\text{m}\,.$$

In this case, the values of θ and p derived in terms of $(d_H - d)$ begin to be inaccurate, as $(d_H - d)$ barely satisfying the condition that it should be $\ll d_H$. On the other hand, we see that η cannot be read to an accuracy of more than 10 % when it is as small as 0.013. When we are dealing with much greater heights, so that d_H is greatly increased, the use of the formulæ in terms of $(d_H - d)$ becomes increasingly important for positions near the horizon. If, for instance, we take $h_1 = 8000\,\mathrm{m}$ and $h_2 = 2000\,\mathrm{m}$, so that d_H is increased to 479 km, then for $\lambda = 1\,\mathrm{cm}$ we have at the limit of the geometric-optical method where $(d_H - d)$ is again 8.02 km, and $d = 471\,\mathrm{km}$ the values

$$p = \frac{\sqrt{8000 \cdot 2000}}{6370 \cdot 479} \cdot (8.02)^2 = 0.084\,32\,\mathrm{m}\,.$$

In this case, we infer from Equation 6.3 that η should be given by

$$\frac{2 \cdot 8000 \cdot 2000}{471} \cdot \eta \cdot 10^{-3} = 0.08432\,,$$

so that for η there results 0.001241, which would be unreadable on Figure 3.4.

From these latter examples, we see that at the limit of the applicability of the geometric-optical method, the path difference p may still be several wave-lengths, so that the position of the first maximum of the interference pattern within the visual range may fall in the transition region embracing the horizon. It then becomes a matter of uncertainty, unless the excessive labor of handling the complete full wave formula in this region is undertaken, what the shape of the interference pattern is within the visual range before the geometric-optical picture becomes valid. It is probable that this picture can be used beyond the condition in Equation 5.3, up to the position of the first maximum predicted by it to give an approximative idea of the true field-strength variations, and this aspect of the problem leads to some interesting analytic relationships between the field just within the horizon given by the geometric-optical treatment, and the diffraction field given by the first term of the full-wave formula. We shall not, however, enter into this matter here, as we have been primarily concerned with the geometry of the problem where the simple ray picture is valid.

Part II.

Second Paper

The Concept of the Equivalent Radius of the Earth in Tropospheric Propagation

Abstract

The concept of the equivalent radius of the earth to take account of a linear variation of refractive index with height in tropospheric refraction is re-examined. It is shown that the transformation is not limited to nearly horizontal rays, but that essentially it reduces the curvature of the earth by that of a ray travelling horizontally and the curvature of the rays by the amount required to straighten them at whatever angle to the horizontal they may be going. The results obtained geometrically in a previous paper for the angle of elevation at the reflection point, the optical path difference between the direct and indirect rays and the divergence factor are derived by simple analysis, affording a useful check on the method.

1. Introduction

In an earlier paper, [23], here referred to as Part I, the author has considered the modification to the results for a flat earth produced by the curvature of the earth in the geometric -optical treatment of ground-wave propagation within the horizon. Many equivalent solutions of this problem have been given, some of which are referred to in the books edited by Burrows and Attwood, [5], and by Kerr, [21].

In this problem it is usual to take into account the effect of the troposphere by assuming that its refractive index decreases linearly with height above the ground and then applying a transformation that straightens the rays and reduces the curvature of the earth accordingly. Schelleng, Burrows and Ferrell, [28], examined the nature of this transformation by using a model for the troposphere for which the ray equation could be solved in exact form and showed that the error involved was negligible provided that the ray paths were nearly horizontal.

It is the purpose of the present paper to examine this matter from first principles and to show more clearly the approximations upon which the transformation is based. The results will be developed in the form used in Part I to show that the method there given of using the equivalent radius of the earth was correct.

2. Radius of Curvature of the Ray Paths

It is convenient here to give a simple derivation of the well-known expression for the curvature of the ray paths in terms of the gradient of refractive index. In Figure 2.1 an element of phase surface FG moves along the ray path into the position $F'G'$. If the phase velocity along the arc FF' is v and that along GG' is $v + \delta v$, then

$$\frac{\delta v}{v} = \frac{\delta R}{R}\,, \tag{2.1}$$

where R is the radius of curvature of the arc FF'.
In terms of the refractive index μ, the phase velocity is given by

$$v = \frac{c}{\mu}\,,$$

where c is the free space velocity, so that

$$\frac{\delta v}{v} = -\frac{\delta \mu}{\mu}\,. \tag{2.2}$$

Thus from Equation 2.1 and Equation 2.2 we have

$$\frac{1}{R} = -\frac{1}{\mu} \cdot \frac{\delta \mu}{\delta R}\,. \tag{2.3}$$

If, as in Figure 2.1, the ray path makes an angle with the surface of constant refractive index, then with reference to Figure 2.1 we have

$$\delta n = \delta R \cdot \cos(\zeta)\,, \tag{2.4}$$

and thus from the combination of Equation 2.3 and Equation 2.4 we get

$$\frac{1}{R} = -\frac{1}{\mu} \cdot \frac{\partial \mu}{\partial n} \cdot \cos(\zeta)\,, \tag{2.5}$$

where $\frac{\partial \mu}{\partial n}$ is the gradient of refractive index at the point or atmospheric layer n under consideration.

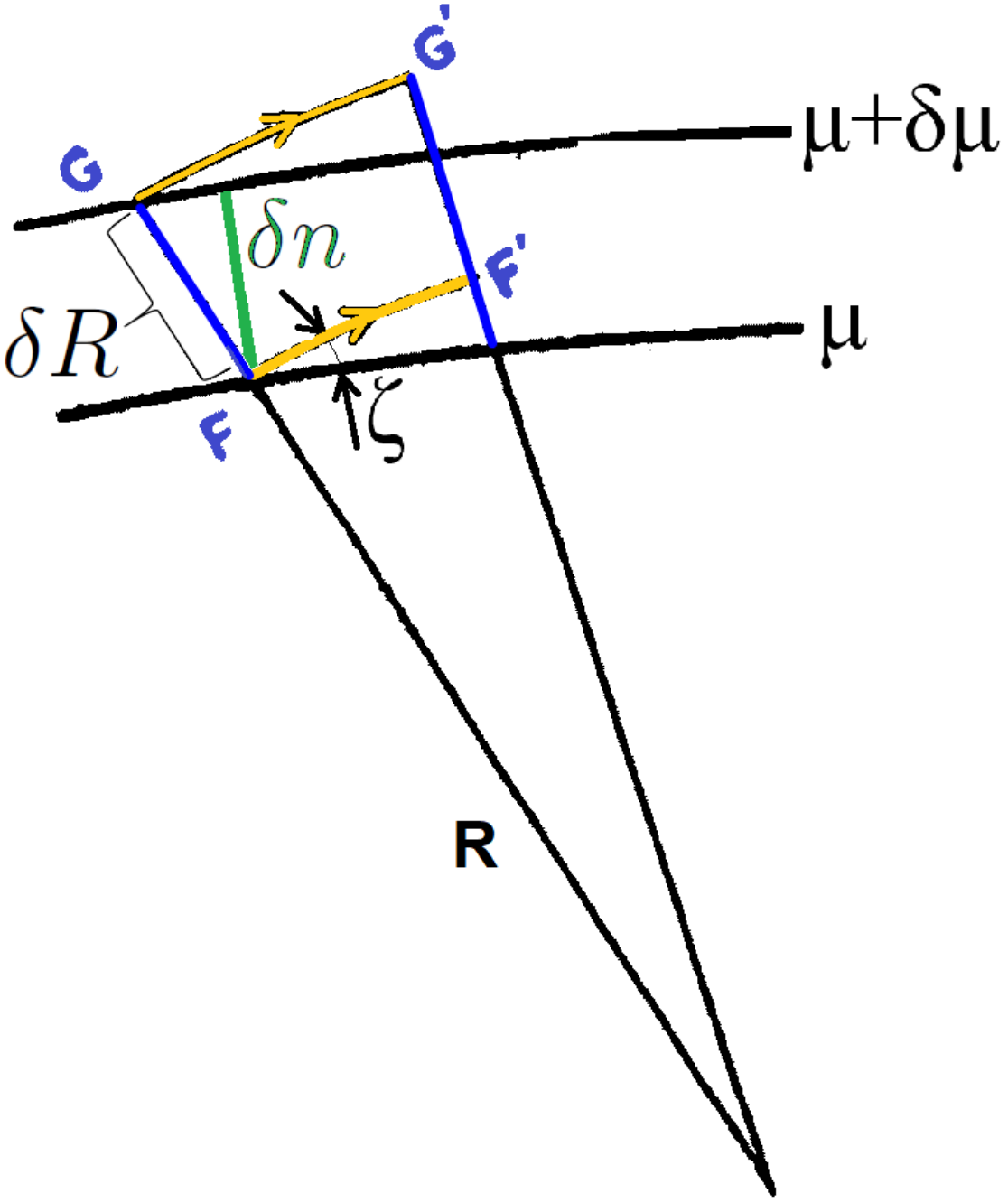

Figure 2.1.: Geometrical illustration of an element FG moving along a ray path over the surface of the earth

3. The Ray Path in the Troposphere

In applying the above results to the troposphere, it is assumed that the surfaces of constant refractive index are spheres concentric with the earth, so that the gradient $\frac{\partial \mu}{\partial n}$ is equal to $\frac{\mathrm{d}\mu}{\mathrm{d}h}$, where h is the height above the ground. Furthermore this gradient is assumed to be constant, so that the refractive index μ as a function of height h is

$$\mu = \mu_0 + \frac{\mathrm{d}\mu}{\mathrm{d}h} \cdot h \,, \tag{3.1}$$

where μ_0 is the (initial) value of the refractive index at the ground.
Actually μ_0 could be taken as unity since it exceeds it by only about 3×10^{-4}, but it will be formally retained for the sake of completeness and because it stresses the importance of the gradient of refractive index even when the overall changes are small.
With this model, Equation 2.5 becomes

$$\frac{1}{R} = -\frac{1}{\mu} \cdot \frac{\mathrm{d}\mu}{\mathrm{d}h} \cdot \cos(\zeta) \,, \tag{3.2}$$

and if R_0 is the radius of curvature of a ray travelling horizontally immediately above the surface of the earth, it follows that

$$\frac{1}{R_0} = -\frac{1}{\mu_0} \cdot \frac{\mathrm{d}\mu}{\mathrm{d}h} \,, \tag{3.3}$$

so that Equation 3.1 may be written in the form

$$\mu = \mu_0 \cdot \left(1 - \frac{h}{R_0}\right) \,. \tag{3.4}$$

Suppose now that in Figure 3.1, PQ is an element of length δp of a ray path extending from A to B and that it subtends an angle $\delta\psi$ at the centre of curvature of the ray path at P. If the corresponding element PM along the horizontal arc through P subtends an angle $\delta\varphi$ at the centre of the earth, then since $\angle QPM = \zeta$,

$$\cos(\zeta) = \frac{PM}{PQ} = \frac{(r_0 + h) \cdot \delta\varphi}{R \cdot \delta\psi} \,,$$

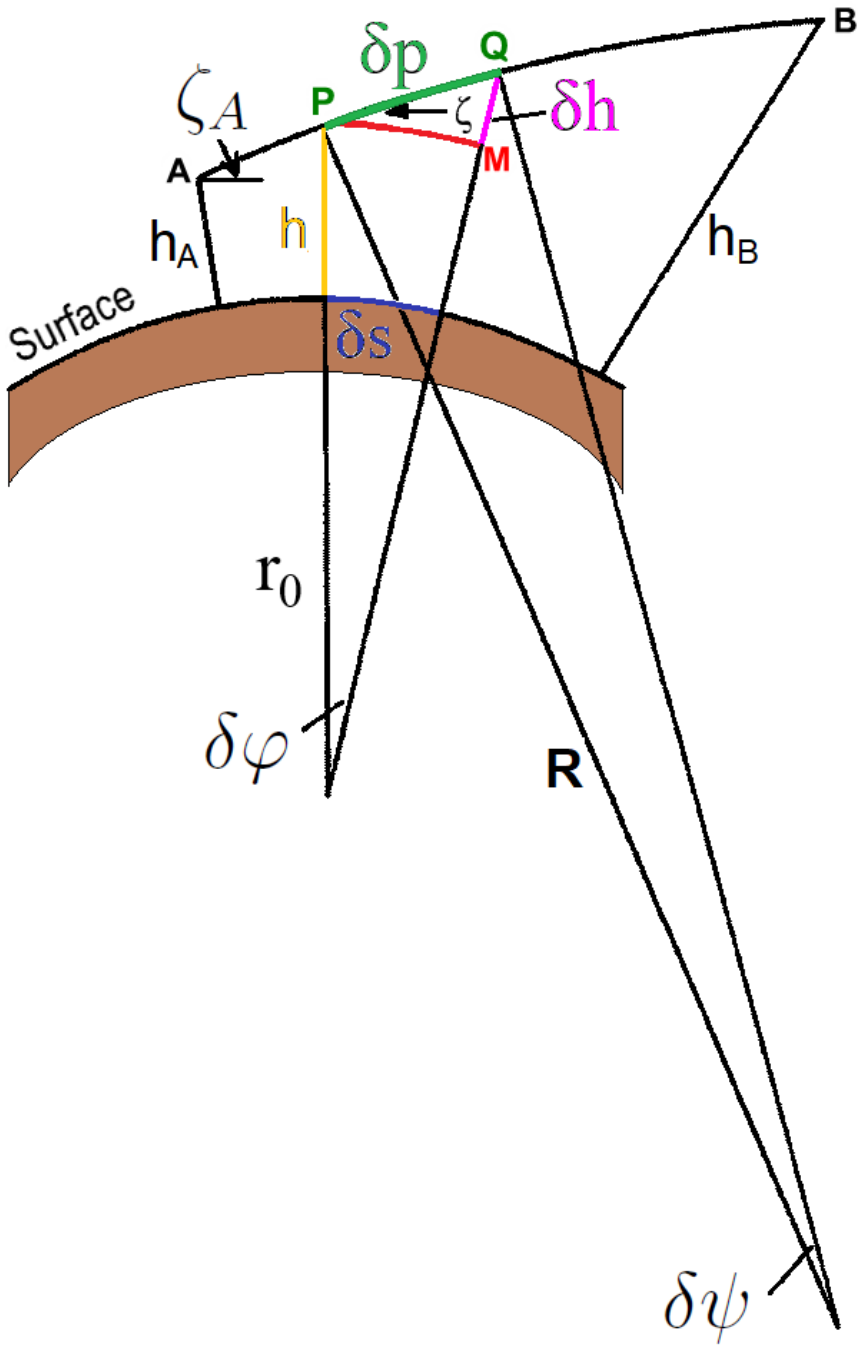

Figure 3.1.: Illustration of determination of the angle ζ that the ray path PQ (green line) in the troposphere makes with the surface of constant refractive index

which when substituted in Equation 3.2 gives

$$\delta\psi = \left(-\frac{1}{\mu}\cdot\frac{\mathrm{d}\mu}{\mathrm{d}h}\right)\cdot(r_0+h)\cdot\delta\varphi\,,$$

i.e., from Equation 3.3

$$\delta\psi = \frac{\mu_0}{\mu}\cdot\frac{(r_0+h)}{R_0}\cdot\delta\varphi\,. \tag{3.5}$$

If δs is the corresponding distance measured over the curved surface of the earth according to

$$\delta s = r_0\cdot\delta\varphi\,, \tag{3.6}$$

then Equation 3.5 may be written in the form

$$\delta\psi = \frac{\mu_0}{\mu}\cdot\left(1+\frac{h}{r_0}\right)\cdot\frac{\delta s}{R_0}\,. \tag{3.7}$$

If the value of ζ at the starting point is ζ_A, then as the tangent to the ray path rotates in space through an angle ψ on passing from A to P while the horizontal direction rotates through an angle φ, it follows that

$$\zeta = \zeta_A + \varphi - \psi\,. \tag{3.8}$$

Also as $MQ = \delta h$,

$$\delta h = (r_0+h)\cdot\tan(\zeta)\cdot\delta\varphi\,,$$

i.e., from Equation 3.6

$$\delta h = \left(1+\frac{h}{r_0}\right)\cdot\tan(\zeta)\cdot\delta s\,, \tag{3.9}$$

while similarly

$$\delta p = \left(1+\frac{h}{r_0}\right)\cdot\sec(\zeta)\cdot\delta s\,. \tag{3.10}$$

These equations are subject to the initial conditions that $h = h_A$ and $\zeta = \zeta_A$ at $s = 0$.

From Equation 3.4, Equation 3.6, Equation 3.7 and Equation 3.8 it follows that

$$\delta\zeta = \left(\frac{1}{r_0}-\frac{1}{R_0}-\frac{2h}{r_0\cdot R_0}\right)\cdot\frac{\delta s}{\left(1-\frac{h}{R_0}\right)}\,. \tag{3.11}$$

Equation 3.9 and Equation 3.11 can be regarded as giving h as a function of s for the given initial conditions. Alternatively, they give ζ_A as a function of h_A and h_B for a ray path which starts at height h_A at $s = 0$ and ends at height h_B at a distance $s = s_0$, say, measured over the surface of the earth.

4. The Equivalent Radius of the Earth

The form of Equation 3.11 suggests the definition of an equivalent radius r_e given by

$$\frac{1}{r_e} = \frac{1}{r_0} - \frac{1}{R_0}\,. \tag{4.1}$$

It is usually assumed that $R_0 = 4 \cdot r_0$, so that $r_e = \frac{4}{3} \cdot r_0$, under normal or standard tropospheric conditions. In this introduction of the equivalent radius of the earth no limitation has been placed on the value of ζ_A, which has not so far been restricted to small values corresponding to nearly horizontal rays.

The validity of the concept of the equivalent radius depends upon being able to express the solution of the ray path equations to a sufficient degree of accuracy by that for a homogeneous atmosphere with r_0 everywhere replaced by r_e. It then becomes possible to solve the problem by using a geometrical transformation in which the earth has an equivalent radius r_e and the ray paths are straight.

In order to see under what restrictions, if any, this becomes possible, the equations will be considered first in rather general terms. The method to be adopted will be to assume that h can be expanded in a power series in s, and a preliminary consideration of the limiting conditions at low and high angles of elevation suggests that it is sufficient to limit the series to a quadratic form, so that

$$h = h_A + a \cdot s + b \cdot s^2\,, \tag{4.2}$$

where the coefficients a and b are determined by substituting Equation 4.2 in Equation 3.9 and Equation 3.11, thereby resulting in Equation 4.2 as the equation of the ray path.

5. Height and Angle Relationships

The integral form of Equation 3.9 is

$$\int_{h_A}^{h} \mathrm{d}h = \int_0^s \left(1 + \frac{h}{r_0}\right) \cdot \tan(\zeta)\, \mathrm{d}s$$

so that from Equation 4.2

$$a \cdot s + b \cdot s^2 = \int_0^s \left(1 + \frac{h}{r_0}\right) \cdot \tan(\zeta)\, \mathrm{d}s\,. \tag{5.1}$$

Thus in order to determine a and b, it is necessary to expand the integrand in Equation 5.1 only to the first power in s. The value of ζ is given from Equation 3.11 and Equation 4.1 by

$$\int_{\zeta_A}^{\zeta} \mathrm{d}\zeta = \int_0^s \left(\frac{1}{r_e} - \frac{2h}{r_0 \cdot R_0}\right) \cdot \frac{1}{\left(1 - \frac{h}{R_0}\right)}\, \mathrm{d}s\,. \tag{5.2}$$

For use in Equation 5.1, the integrand on the right-hand side of Equation 5.2 needs only to be expanded as far as the constant term, so that from Equation 4.2, Equation 5.2 results in

$$\zeta = \zeta_A + c \cdot s\,, \tag{5.3}$$

where

$$c = \frac{\left(\frac{1}{r_e} - \frac{2h_A}{r_0 \cdot R_0}\right)}{\left(1 - \frac{h_A}{R_0}\right)}\,. \tag{5.4}$$

By taking the first two terms of a Taylor expansion, Equation 5.3 gives

$$\tan(\zeta) = \tan(\zeta_A) + c \cdot s \cdot \sec^2(\zeta_A)\,. \tag{5.5}$$

Thus from Equation 4.2 and Equation 5.5, to the first power in s this gives

$$\left(1 + \frac{h}{r_0}\right) \cdot \tan(\zeta) = \left(1 + \frac{h_A}{r_0}\right) \cdot \tan(\zeta_A) + \left[\frac{a}{r_0} \cdot \tan(\zeta_A) + \left(1 + \frac{h_A}{r_0}\right) \cdot c \cdot \sec^2(\zeta_A)\right] \cdot s\,,$$

so that on integration of Equation 5.1 the two constants a and b are given by

$$a = \left(1 + \frac{h_A}{r_0}\right) \cdot \tan(\zeta_A) \tag{5.6}$$

and

$$b = \frac{1}{2} \cdot \left[\frac{a}{r_0} \cdot \tan(\zeta_A) + \left(1 + \frac{h_A}{r_0}\right) \cdot c \cdot \sec^2(\zeta_A)\right] ,$$

and so, by means of Equation 5.6 the constant b is given by

$$b = \frac{1}{2} \cdot \left(1 + \frac{h_A}{r_0}\right) \cdot \left[\frac{1}{r_0} \cdot \tan^2(\zeta_A) + c \cdot \sec^2(\zeta_A)\right] . \tag{5.7}$$

In order to establish the equivalence theorem, it is now necessary to invoke the condition that the heights considered in tropospheric propagation are very small compared with the radius of the earth, so that Equation 5.4 gives

$$c = \frac{1}{r_e} \tag{5.8}$$

and hence Equation 5.6 and Equation 5.7 become

$$a = \tan(\zeta_A) \tag{5.9a}$$

$$b = \frac{1}{2} \cdot \left[\frac{1}{r_e} + \left(\frac{1}{r_0} + \frac{1}{r_e}\right) \cdot \tan^2(\zeta_A)\right] . \tag{5.9b}$$

Thus from Equation 4.2, Equation 5.9a and Equation 5.9b it follows that

$$h = h_A + s \cdot \tan(\zeta_A) + \frac{s^2}{2 \cdot r_e} \cdot \left[1 + \left(1 + \frac{r_e}{r_0}\right) \cdot \tan^2(\zeta_A)\right] . \tag{5.10}$$

At first sight it appears that for large values of $\tan(\zeta_A)$ the power series for h diverges, but for a given value of $(h - h_A)$, s decreases as $\tan(\zeta_A)$ increases and their product approaches the flat earth limit of $(h - h_A)$. On the other hand when $\tan(\zeta_A)$ is sufficiently small, $s \cdot \tan(\zeta_A)$ becomes small compared with $\frac{s^2}{2 \cdot r_e}$.
If

$$s \cdot \tan(\zeta_A) = (h - h_A) \cdot (1 - \alpha) \tag{5.11}$$

and

$$\frac{s^2}{2 \cdot r_e} = (h - h_A) \cdot (1 - \beta) \tag{5.12}$$

then as

$$\zeta_A \to 0\,, \quad \alpha \to 1 \text{ and } \beta \to 0\,,$$

and as

$$\zeta_A \to \frac{\pi}{2}\,, \quad \alpha \to 0 \text{ and } \beta \to 1\,.$$

On substituting Equation 5.11 and Equation 5.12 in Equation 5.10 and dividing through by $(h - h_A)$, it will be found that

$$\alpha + \beta = 1 + \frac{h - h_A}{2 \cdot r_e} \cdot \left(1 + \frac{r_e}{r_0}\right) \cdot (1 - \alpha)^2\,. \tag{5.13}$$

Now from Equation 5.11 and Equation 5.12 it follows that

$$s \cdot \tan(\zeta_A) + \frac{s^2}{2 \cdot r_e} = (h - h_A) \cdot (2 - \alpha - \beta)\,,$$

i.e., from Equation 5.13

$$s \cdot \tan(\zeta_A) + \frac{s^2}{2 \cdot r_e} = (h - h_A) \cdot \left[1 - \mathcal{O}\left(\frac{h - h_A}{r_e}\right)\right]\,.$$

Thus over the whole range of ζ_A Equation 5.10 may be replaced to the order of accuracy under consideration by

$$h = h_A + s \cdot \tan(\zeta_A) + \frac{s^2}{2 \cdot r_e}\,, \tag{5.14}$$

in which the s^2 term is of course already negligible under conditions in which $\tan^2(\zeta_A)$ is not $\ll 1$.

The fact that $s \cdot \tan(\zeta_A)$ approaches the limit $(h - h_A)$ as $\zeta_A \to \frac{\pi}{2}$ assures the convergence of the power series for h and justifies the restriction to the quadratic form in Equation 4.2.

It follows from Equation 5.14 that for $h = h_B$ and $s = s_0$ the following holds:

$$\tan(\zeta_A) = \frac{h_B - h_A}{s_0} - \frac{s_0}{r_e}\,. \tag{5.15}$$

In Equation 5.14 and Equation 5.15, r_0 does not appear explicitly, and if the case for a homogeneous atmosphere is derived by putting $R_0 = \infty$, r_e in these equations is replaced by r_0. It follows that they have the same form that they would have for a homogeneous atmosphere above an earth of radius r_e, and the equivalence theorem is established as far as the height and angle relationship is concerned. Using the geometrical transformation in which the ray paths are straight and the earth has an equivalent radius r_e, Equation 5.15 may be derived by simple geometry along the lines of the proof given in Part I. It is interesting to note that for the

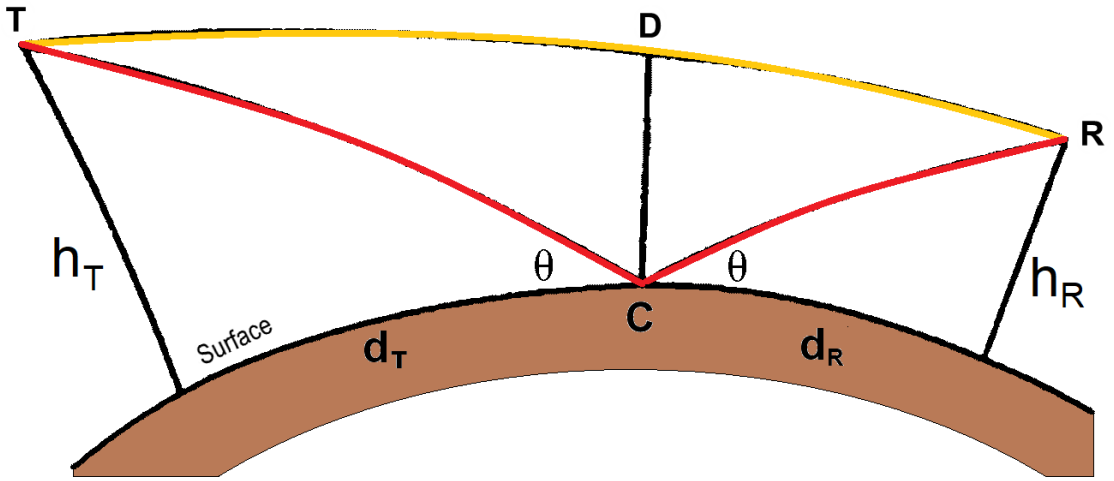

Figure 5.1.: Illustration of ray paths TC and CR (red line) and direct path TR (yellow line) with reflection point C for the indirect ray as discussed in the text

case when $h_A = 0$, i.e., for a ray starting at the surface of the earth where ζ_A is equal, say, to θ, the geometrical proof leads formally to the expansion

$$h = s \cdot \tan(\theta) + \frac{s^2}{2 \cdot r_e} \cdot \left[1 + 2 \cdot \tan^2(\theta)\right] , \tag{5.16}$$

which confirms the general form of Equation 5.10, though it does not of course affect the validity of the approximation in Equation 5.14 upon which the equivalence theorem is based.

Consider now the ray paths shown in Figure 5.1 between a transmitter T at a height h_T and a receiver R at a height h_R and a distance d away measured over the surface of the earth. Let the reflection point for the indirect ray be C, which divides d into the parts d_T and d_R for the ray paths TC and CR, so that

$$d = d_T + d_R \, . \tag{5.17}$$

Suppose that the angle of elevation at C is θ, which by the law of reflection is the same for both paths, regarding the path TC in the reverse sense from C to T for convenience.
Equation 5.15 may be applied to the path CT by putting $\zeta_A = 0$, $h_A = 0$, $h_B = h_T$ and $s_0 = d_T$, so that

$$\tan(\theta) = \frac{h_T}{d_T} - \frac{d_T}{2 \cdot r_e} \tag{5.18}$$

and similarly for the path CR

$$\tan(\theta) = \frac{h_R}{d_R} - \frac{d_R}{2 \cdot r_e} \,. \tag{5.19}$$

By using Equation 5.17, Equation 5.18 and Equation 5.19 may be solved for d_T (or d_R) and $\tan(\theta)$, giving the angle of elevation and the position of the reflection point. The cubic equation obtained in the process may be solved graphically exactly as in Part I using the charts there given.

In the transformed picture the same actual heights and distance d apart of the transmitter and receiver are used, and the position of the reflection point and the value of the angle of elevation are the same as in the actual situation in Figure 5.1. This is of practical importance where in reality ground irregularities may exist and it is necessary to see whether the reflection point given by the idealized smooth earth picture is affected by any irregularities in the region.

6. The Path Difference Formula

The phase difference between the signals at R arriving by the direct path TR in Figure 5.1 and by reflection at C depends upon the difference of the corresponding optical paths. For small values of elevation this difference is very small compared with the length of the individual paths and it is not legitimate to neglect small terms in the integrals without careful consideration.
The optical path p' is given by

$$p' = \int \mu \, \mathrm{d}p \tag{6.1}$$

taken between appropriate limits.
From Equation 3.4 and Equation 3.10, Equation 6.1 may be written in the form

$$p' = \int_0^s \mu_0 \cdot \left(1 - \frac{h}{R_0}\right) \cdot \left(1 + \frac{h}{r_0}\right) \cdot \sec(\zeta) \, \mathrm{d}s \,. \tag{6.2}$$

An attempt to expand the integrand as a power series in s as far as the s^2 term, so that p' is given to the s^3 term required for the differencing process, is laborious. The essential condition is now to retain terms in the integrand of order $\frac{h}{r_0}$ while neglecting the extremely small terms of order $\frac{h^2}{r_0^2}$. Thus it is permissible with the use of Equation 4.1 to write

$$\left(1 - \frac{h}{R_0}\right) \cdot \left(1 + \frac{h}{r_0}\right) = 1 + \frac{h}{r_e} \,. \tag{6.3}$$

In this equation h can be replaced by its value in Equation 5.14, since the approximations made in deriving it from Equation 3.9 correspond to the neglection of terms of order $\frac{h^2}{r_0^2}$ in Equation 6.3.

Returning to Equation 5.2, by the mean value theorem it may be written as

$$\int_{\zeta_A}^{\zeta} \mathrm{d}\zeta = \int_0^s \frac{1}{r_e} \cdot (1 - \gamma) \, \mathrm{d}s \,,$$

where γ is a constant of the order of $\frac{h}{R_0}$, so that

$$\zeta = \zeta_A + \frac{1}{r_e} \cdot (1 - \gamma) \cdot s \,.$$

The TAYLOR expansion for $\sec(\zeta)$ as far as the s^2 term is thus given by

$$\begin{aligned} \sec(\zeta) = \sec(\zeta_A) &+ \sec(\zeta_A) \cdot \tan(\zeta_A) \cdot \frac{(1-\gamma)}{r_e} \cdot s \\ &+ \sec(\zeta_A) \cdot \left[1 + 2 \cdot \tan^2(\zeta_A)\right] \cdot \frac{(1-\gamma)^2}{2 \cdot r_e^2} \cdot s^2 \,. \end{aligned}$$

By an argument similar to that used above, remembering that γ is of the order of $\frac{h}{R_0}$, the neglection of terms of order $\frac{h^2}{r_0^2}$ leads to

$$\sec(\zeta) = \sec(\zeta_A) \cdot \left(1 + \frac{s}{r_e} \cdot \tan(\zeta_A) + \frac{s^2}{2 \cdot r_e^2}\right) \,,$$

which from Equation 5.14 may be put in the form

$$\sec(\zeta) = \sec(\zeta_A) \cdot \left(1 + \frac{h - h_A}{r_e}\right) \,.$$

With Equation 6.3 this gives

$$\left(1 - \frac{h}{R_0}\right) \cdot \left(1 + \frac{h}{r_0}\right) \cdot \sec(\zeta) = \sec(\zeta_A) \cdot \left(1 + \frac{2h - h_A}{r_e}\right) \,,$$

so that Equation 6.2 becomes

$$p' = \mu_0 \cdot \sec(\zeta_A) \cdot \int_0^s \left(1 + \frac{2h - h_A}{r_e}\right) \mathrm{d}s \,, \tag{6.4}$$

where h is given by Equation 5.14 and ζ_A by Equation 5.15.

In view of the equivalence theorem already established for h and ζ_A, it follows from Equation 6.4 that the equivalence also holds for the optical path difference, replacing the atmosphere by one which is homogeneous with a refractive index μ_0 and the earth by a sphere of radius r_e, without any formal restriction to small values of ζ_A.
However, at high angles of elevation the path difference between the direct and indirect rays is no longer very small compared with the lengths of the individual paths, since the distance along the ground between the transmitter and receiver is then only comparable with their heights above the surface. Moreover, the ground irregularities make a precise calculation of the path difference at large angles of ele-

vation based on an ideal theory unwarranted, especially where the terminal heights are large compared with the wavelength and the path difference may contain many wavelengths. It is only at small angles of elevation in the study of the interference pattern near the horizon that the problem is of practical importance.
In any case the direct integration of Equation 6.4 using Equation 5.14 and Equation 5.15 would lead to a cumbersome result and it would be simpler to resort to a geometrical solution based on the equivalence theorem. However, it is interesting as a check on the analytical approach adopted here to derive the form of the result given in Part I for small angles of elevation.

In Equation 6.4, h is given its value in Equation 5.14 with $\tan(\zeta_A)$ put equal to ζ_A and $\sec(\zeta_A)$ is replaced by $1 + \frac{\zeta_A^2}{2}$, so that

$$p' = \mu_0 \cdot \int_0^s \left(1 + \frac{\zeta_A^2}{2}\right) \cdot \left(1 + \frac{h_A}{r_e} + \frac{2\zeta_A s}{r_e} + \frac{s^2}{r_e^2}\right) \mathrm{d}s\,,$$

which can be written for the small values of ζ_A under consideration as

$$p' = \mu_0 \cdot \int_0^s \left(1 + \frac{h_A}{r_e} + \frac{s^2}{r_e^2} + \frac{\zeta_A^2}{2} + \frac{2\zeta_A s}{r_e}\right) \mathrm{d}s\,. \tag{6.5}$$

This result can be obtained more simply by approximating directly to $\sec(\zeta)$ in Equation 6.2 by $\left(1 + \frac{\zeta^2}{2}\right)$, where ζ is given from Equation 5.3 and Equation 5.8 by $\left(\zeta_A + \frac{s}{r_0}\right)$, and again using Equation 6.3 and Equation 5.14.
Referring back to Figure 5.1 it is useful to divide the path TR into two parts TD and DR, where D is at a distance d_T from T measured over the surface of the earth. The optical path difference corresponding to the path $TC + CR - TR$ may then be split up into $TC - TD$ and $RC - RD$. For TC and TD the upper limit in both cases for the integral in Equation 6.5 is d_T, and it leads to a considerable simplification to do part of the differencing in the integrands before integration. For both paths the starting point is T, so that h_A is the same for both, namely h_T. Thus in Equation 6.5, $\left(1 + \frac{h_A}{r_e} + \frac{s^2}{r_e^2}\right)$ is common to both integrands, so that it is only necessary to consider the difference between integrals of the form

$$\mathcal{I} = \mu_0 \cdot \int_0^{d_T} \left(\frac{\zeta_A^2}{2} + \frac{2\zeta_A s}{r_e}\right) \mathrm{d}s = \mu_0 \cdot \frac{d_T}{2} \cdot \left(\zeta_A^2 + 2\zeta_A \cdot \frac{d_T}{r_e}\right)\,. \tag{6.6}$$

Now in considering the path TC in the opposite direction, it has been seen that at C, $\zeta_A = \theta$, and thus from Equation 5.3 and Equation 5.8 it is $\zeta = \theta + \frac{d_T}{r_0}$ at T. It follows that in the direction TC,

$$\zeta_A = -\left(\theta + \frac{d_T}{r_e}\right)$$

at T, and so from Equation 6.6 on reduction

$$\mathcal{I}_{TC} = \mu_0 \cdot \frac{d_T}{2} \cdot \left(\theta^2 - \frac{d_T^2}{r_e^2} \right) . \tag{6.7}$$

Suppose for the moment that, for the path TD, the value of ζ_A at T is ζ_T, then from Equation 6.6 it follows that

$$\mathcal{I}_{TD} = \mu_0 \cdot \frac{d_T}{2} \cdot \left(\zeta_T^2 + 2\zeta_T \cdot \frac{d_T}{r_e} \right) . \tag{6.8}$$

The required optical path difference for $TC - TD$ is from Equation 6.7 and Equation 6.8 given by

$$\mathcal{I}_{TC} - \mathcal{I}_{TD} = \mu_0 \cdot \frac{d_T}{2} \cdot \left[\theta^2 - \left(\zeta_T + \frac{d_T}{r_e} \right)^2 \right] . \tag{6.9}$$

The value of ζ_T may be found from Equation 5.15 for the path TR by putting $\tan(\zeta_A) = \zeta_T$, $h_A = h_T$, $h_B = h_R$ and $s_0 = d$, so that

$$\zeta_T = \frac{h_R - h_T}{d} - \frac{d}{2 \cdot r_e} , \tag{6.10}$$

where d is the total distance between T and R, as given in Equation 5.17.
Now from Equation 5.18 and Equation 5.19 for small values of θ it follows that

$$h_T = d_T \cdot \theta + \frac{d_T^2}{2 \cdot r_e} \tag{6.11}$$

and

$$h_R = d_R \cdot \theta + \frac{d_R^2}{2 \cdot r_e} . \tag{6.12}$$

Thus from Equation 6.10, Equation 6.11, Equation 6.12 and Equation 5.17 on reduction we get

$$\zeta_T = \frac{\theta \cdot (d_R - d_T)}{d} - \frac{d_T}{r_e}$$

and hence

$$\zeta_T + \frac{d_T}{r_e} = \frac{\theta \cdot (d_R - d_T)}{d} . \tag{6.13}$$

It follows from Equation 5.17 and Equation 6.13 that

$$\theta^2 - \left(\zeta_T + \frac{d_T}{r_e} \right)^2 = 4 \cdot \frac{d_T \cdot d_R}{d^2} \cdot \theta^2 , \tag{6.14}$$

and hence from Equation 6.9 and Equation 6.14 that

$$\mathcal{I}_{TC} - \mathcal{I}_{TD} = 2\mu_0 \cdot \frac{d_T^2 \cdot d_R}{d^2} \cdot \theta^2 \,. \tag{6.15}$$

The corresponding value for the path difference $RC - RD$ can be obtained immediately from Equation 6.15 by interchanging d_T and d_R, giving

$$\mathcal{I}_{RC} - \mathcal{I}_{RD} = 2\mu_0 \cdot \frac{d_R^2 \cdot d_T}{d^2} \cdot \theta^2 \,. \tag{6.16}$$

Finally by adding together Equation 6.15 and Equation 6.16, the optical path difference, say $\Delta p'$, is given by

$$\Delta p' = 2\mu_0 \cdot \frac{d_T \cdot d_R}{d} \cdot \theta^2 \,. \tag{6.17}$$

This value agrees with that given in Part I(see Equation 2.2), where μ_0 was taken as unity, and leads to a similar technique for expressing $\Delta p'$ as the flat-earth value multiplied by a factor that can be found from the graphs in Part I.

A point of some interest arises by returning to Equation 6.2 and resolving it as follows:–
We have

$$p' = \mu_0 \cdot (A - B) \tag{6.18}$$

where

$$A \equiv \int_0^s \left(1 + \frac{h}{r_0}\right) \cdot \sec\zeta \, \mathrm{d}s \tag{6.19}$$

and

$$B \equiv \int_0^s \frac{h}{R_0} \cdot \left(1 + \frac{h}{r_0}\right) \cdot \sec\zeta \, \mathrm{d}s \,. \tag{6.20}$$

From Equation 3.10 and Equation 6.19 it follows that

$$A = p \tag{6.21}$$

while

$$B = \int_0^s \frac{h}{R_0} \, \mathrm{d}s \,, \tag{6.22}$$

where in Equation 6.21 p is the actual geometrical path length, while in Equation 6.22 the higher order terms in Equation 6.20 are rejected.

Now A is very much greater than B, and as it takes account of the fact that the ray path is not straight but is a circular arc, it might be thought at first sight that it contains the significant part of the correction for the gradient of refractive index. It turns out, however, that although the individual paths given by B in Equation 6.22 are very small, the relevant path difference contributed by B is

comparable with that derived from A.

In fact, by a geometrical argument based on the assumption that the paths are circular arcs of radius R_0, it can be shown that the difference between the A-terms is ΔA, where

$$\Delta A = \frac{2 \cdot d_T \cdot d_R}{d} \cdot \theta \cdot \left(\theta - \frac{\psi}{2} \right) , \tag{6.23}$$

and where ψ is the angle subtended by the arc TR at its centre of curvature. Furthermore, from the integration of Equation 6.22, using the value of h in Equation 5.14 for small ζ_A, it can be shown that the corresponding value of ΔB is

$$\Delta B = -\frac{d_T \cdot d_R}{d} \cdot \theta \cdot \psi \, , \tag{6.24}$$

which with Equation 6.23 leads from Equation 6.18 to Equation 6.17.

Equation 6.23 shows that on the horizon, where $\theta = 0$, the geometrical path difference is zero, as is obvious, but that as θ is increased from zero, the path difference becomes negative and remains so until $\theta = \psi/2$. This means that there is an initial range of values for which the actual curved path for the direct ray is longer than the sum of the two components of the indirect ray path.
The limiting case for $\theta = \psi/2$ is shown in Figure 6.1 for the symmetrical situation in which $h_T = h_R$ and the line CD is bisected by the chord TR at E. It is obvious that the chord TD is equal to the chord TC, and as the arcs have the same radius R_0, they must also be equal. Thus, the total arc TR is equal to the sum of the arcs TC and CR, so that the path difference is zero. The chord CT makes an angle $(\theta - \psi/2)$ with the tangent plane through C, where ψ_T is the angle subtended by the arc TC at its centre of curvature. This angle is equal to $\angle ETC = \angle DTE$. As $\psi_T = \psi/2$, it follows that $\angle DTE = \theta - \psi/4$, but this angle is half of the angle $\psi/2$ subtended by DR at the centre of curvature of the arc TDR, so that $\theta - \psi/4 = \psi/4$ and $\theta = \psi/2$ as anticipated.

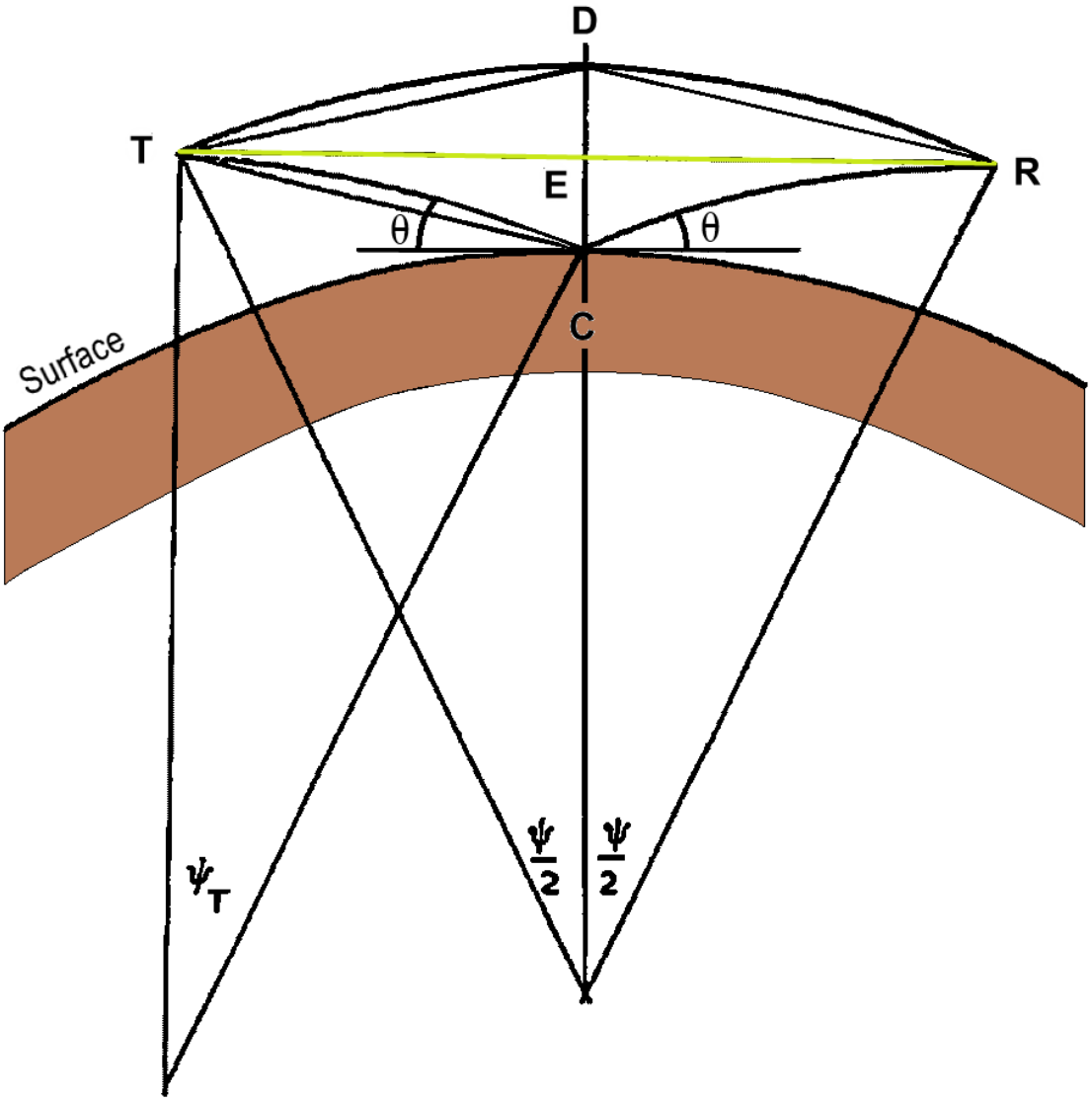

Figure 6.1.: Limiting case for the angle of elevation $\theta = \psi/2$

7. The Divergence Factor

There remains the consideration of the divergence factor. The argument in this case is essentially a geometrical one and all that really needs to be said is that in applying the above analysis the equivalent radius of the earth everywhere replaces the true value. The problem can thus be solved using straight ray paths as in Part I. However, it is instructive to use the above equations to obtain an alternative derivation of the divergence factor $\mathcal{D}$.
In Figure 7.1, in addition to the indirect ray path TC and CR with the reflection point at C, there is shown a neighboring ray that leaves the transmitter at an angle τ below TC and strikes the earth at C' at a distance ϵ ahead of C. The reflected ray then passes through R' at a height δh_R above R. The quantities τ, ϵ and δh_R are limitingly small but have been greatly exaggerated in Figure 7.1 for the sake of clearness.

In free space the corresponding width of the beam of angle τ at a distance d would be $\mathrm{d}\tau$, and as there is no lateral deviation of the rays, the divergence factor $\mathcal{D}$ expressed in terms of relative amplitudes is

$$\mathcal{D} = \sqrt{\frac{\mathrm{d}\tau}{\delta h_R}} \tag{7.1}$$

restricting the argument to small values of the reflection angle θ.
Now for the ray TC, it has been seen that the elevation of the ray at T looking away from C is

$$\zeta = \theta + \frac{d_T}{r_e}\,. \tag{7.2}$$

A neighbouring ray will be defined from Equation 7.2 by

$$\delta\zeta = \delta\theta + \frac{\delta d_T}{r_e} \tag{7.3}$$

and for the particular ray TC', $\delta\zeta = \tau$ and $\delta d_T = -\epsilon$, so that from Equation 7.3 it follows that

$$\tau = \delta\theta - \frac{\epsilon}{r_e}\,. \tag{7.4}$$

The appropriate value of $\delta\theta$ is decided by the fact that both rays pass through T,

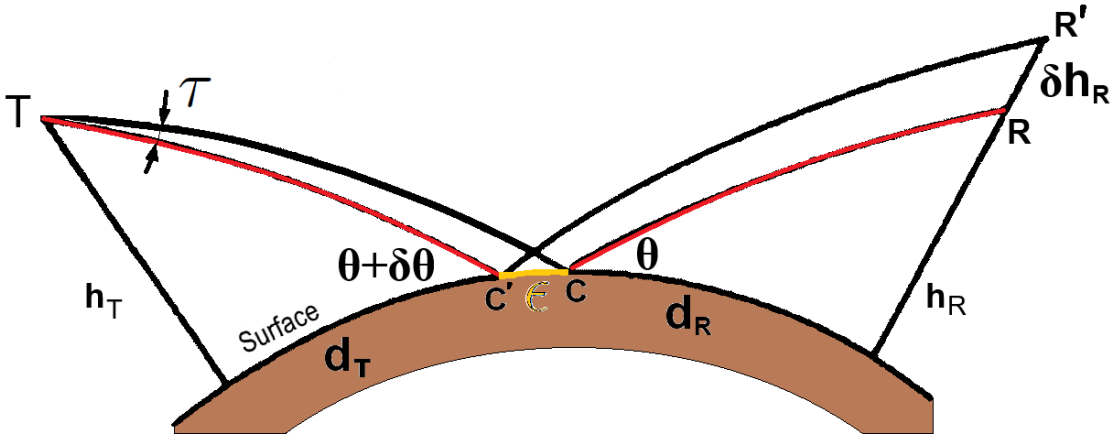

Figure 7.1.: Illustrative derivation of the divergence factor D with neighboring ray (red line) that leaves transmitter T at an angle τ below TC

so that $\delta h_T = 0$, and from Equation 6.11 it follows that

$$0 = d_T \cdot \delta\theta + \theta \cdot \delta d_T + \frac{d_T}{r_e} \cdot \delta d_T \,.$$

Putting $\delta d_T = -\epsilon$, this gives

$$\delta\theta = \left(\frac{\theta}{d_T} + \frac{1}{r_e} \right) \cdot \epsilon \,. \tag{7.5}$$

Thus from Equation 7.4 and Equation 7.5 there results

$$\tau = \frac{\theta \cdot \epsilon}{d_T} \,. \tag{7.6}$$

From Equation 6.12, δh_R is given by

$$\delta h_R = d_R \cdot \delta\theta + \theta \cdot \delta d_R + \frac{d_R}{r_e} \cdot \delta d_R \,.$$

For the ray $C'R'$, $\delta\theta$ is the same as in Equation 7.5 and $\delta d_R = \epsilon$, so that

$$\delta h_R = d_R \cdot \left(\frac{\theta}{d_T} + \frac{1}{r_e} \right) \cdot \epsilon + \theta \cdot \epsilon + \frac{d_R \cdot \epsilon}{r_e} \,,$$

which from Equation 5.17 may be written in the form

$$\delta h_R = \frac{d \cdot \theta \cdot \epsilon}{d_T} + \frac{2 \cdot d_R}{r_e} \cdot \epsilon \,. \tag{7.7}$$

Thus from Equation 7.1, Equation 7.6 and Equation 7.7 we have

$$\mathcal{D} = \frac{1}{\sqrt{1 + \frac{2 \cdot d_T \cdot d_R}{r_e \cdot d \cdot \theta}}} \,. \tag{7.8}$$

Equation 7.8 is equivalent to the value given in Part I with r_0 replaced by r_e and leads to the graphical method there derived for finding $\mathcal{D}$.
If this proof is generalized to include all values of θ, great care has to be taken to include all terms of order $\left(\frac{d}{r_e} \cdot \tan(\theta)\right)$ relative to unity. In Equation 7.1, d in the numerator must be replaced by a more accurate value of the length of the path $TC + CR$, and δh_R in the denominator must be multiplied by $\cos(\zeta_R)$, where ζ_R is the elevation of the ray CR at R, to take account of the fact that the normal to the ray makes an angle ζ_R with the vertical at R.
Moreover, in differentiating the expressions for h_T and h_R to find $\delta\theta$ and δh_R in terms of ϵ, equations of the form in Equation 5.16 must now be used. The final result is

$$\mathcal{D} = \frac{1}{\sqrt{1 + \frac{2 \cdot d_T \cdot d_R \cdot \sec^2(\theta)}{r_e \cdot d \cdot \tan(\theta)}}} \,. \tag{7.9}$$

This result may be obtained by a simple generalization of the geometrical proof given in Part I (see Equation 4.5).

When θ is small, Equation 7.9 degenerates to Equation 7.8, and when $\tan(\theta)$ is large it becomes

$$\mathcal{D} = \frac{1}{\sqrt{1 + \frac{2 \cdot d_T \cdot d_R \cdot \tan(\theta)}{r_e \cdot d}}} \,. \tag{7.10}$$

Under the latter conditions $h_T \coloneqq d_T \cdot \tan(\theta)$ and $h_R \coloneqq d_R \tan(\theta)$, so that

$$\mathcal{D} = \frac{1}{\sqrt{1 + \frac{2 \cdot h_T \cdot h_R}{r_e \cdot (h_T + h_R)}}} \,. \tag{7.11}$$

This result may be obtained by a simple geometrical proof for vertical incidence on a sphere of radius r_e.
The use of Equation 5.16 instead of Equation 5.10, which still contains r_0 explicitly, suggests that in Equation 7.9 to Equation 7.11 the use of r_e is not strictly correct. The point is, however, purely academic, since as θ is increased, $\mathcal{D}$ given by Equation 7.8 becomes effectively unity while the approximation in Equation 7.8 is still amply good. In fact, where expressions of the form in Equation 7.9 and Equation 7.10 have to be used in principle, $\mathcal{D}$ is always very close to unity. This is borne out by Equation 7.11 where the difference of $\mathcal{D}$ from unity is clearly of the order of h/r_e, where h is the smaller of h_T and h_R and is therefore negligible to the order of accuracy assumed throughout.

8. Conclusion

Although the results for small angles of elevation are well known, the presentation adopted has aimed at showing more clearly the nature of the approximations made in the analysis and the range of their validity. It will be noted that throughout the argument h_T and h_R are the actual heights above the true earth and not the equivalent heights above the tangent plane at the reflection point. Similarly, d_T and d_R are actual distances along the true earth.

In the transformed geometry, h_T and h_R are unaltered and become heights above the equivalent earth's surface drawn through the reflection point, which is positioned so that d_T and d_R remain unaltered although measured along the surface of the equivalent earth.

The angle of elevation at the reflection point derived from the transformed geometry is the true angle made by the curved ray paths with the actual earth's surface, which has to be used in finding the FRESNEL reflection coefficient where it is significantly different from -1.

The only essential restriction imposed on the system is that the heights above earth are very small compared with the radius of the earth, a condition that is amply fulfilled for tropospheric propagation when a linear variation of refractive index with height can be assumed. The transformation is not formally confined to nearly horizontal rays, the true nature of it being to straighten the ray paths, whatever their direction relative to the horizontal, and hence whatever their curvature, may be. This generalization arises from the fact that in deriving Equation 3.7 with the help of Equation 3.2 and Equation 3.3, the factor $\cos(\zeta)$ cancels out.

This investigation was prompted partly by the consideration of a transformation given in the Introduction to the new ground-wave propagation curves recently published by the C.C.I.R., [20], in which modified values of the dielectric constant and the conductivity of the earth are used in the geometric-optical formulæ within the horizon.

These values result from a transformation that uses the true radius of the earth and is expressed in terms of reduced heights and distances designed to connect with the reduced values used in the diffraction region. This use of such modified quantities, which are functions of the wave frequency, in the geometric-optical formulæ is presumably equivalent mathematically to the use of the equivalent earth radius concept, but it lacks its physical significance.

Such a transformation is probably more convenient when computing ground-wave curves very accurately near the horizon to join up with the curves obtained by taking a sufficiently large number of terms of the residue series used in the diffraction

region. It is difficult, however, to see how such a transformation can be derived from a purely geometric-optical argument without recourse to a comparison with the form of the modification produced by the gradient of refractive index in the diffraction analysis.
The above treatment is a reassurance that in the geometric-optical region the usual method of transforming the geometry is essentially correct and general in its application, and is preferable for use where this region alone is under consideration.

Part III.

Third Paper

Ground-Wave Propagation over an Inhomogeneous Smooth Earth. Part I. Theoretical Considerations

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Abstract

The problem of ground-wave propagation over an inhomogeneous smooth earth is discussed in terms of the known solution for a homogeneous earth. The inhomogeneity refers only to changes in the earth constants from place to place, and the problem is idealized by assuming a wave radiated from a vertical dipole over a series of homogeneous annular sections. After a statement of some of the conditions the solution would be expected to obey, including the essential one of reciprocity, some fundamental results of the theory for a homogeneous earth are described in a form directly useful to the argument. The solution is first given for the short-wave limit, where it is complete except in the neighbourhood of a boundary. By an approximate consideration of the energy flow at different heights above the ground, the solution is extended to the case of intermediate wavelengths where the first and last boundaries are in the diffraction region of the transmitting and receiving points respectively. It is then shown that a well-known empirical method yields the same solution when it is made reciprocal by taking the geometric mean of the value it gives and the value that would be obtained with the transmitter and receiver interchanged. This method is formally used to obtain a tentative solution for the effect of the disturbance function in the neighbourhood of a boundary. It leads to the striking suggestion that on passing from a section of one value of conductivity to another of a higher one, there is a recovery in field-strength before the attenuation of the wave becomes characteristic of the new section. On crossing the boundary in the other direction, there is a correspondingly increased drop in field-strength before the attenuation takes its new characteristic type. Owing to the lack of sufficiently controlled conditions, most of the existing experimental results are inconclusive with regard to these features at a boundary, but some evidence is given in support of them. Stress is laid on the need for further experiments specifically designed to study the field near a land-sea boundary. The paper deals briefly with the practical application of the method, and gives a specimen field-strength/distance curve for a route consisting of several land and sea sections. It concludes by pointing out that further research is needed, especially with regard to the phase relationships, as the argument has dealt only with field-strength values.

Preliminary Notes

In the following, the unit e.m.u. and e.s.u., i.e. electromagnetic units and electrostatic units, respectively, are used for permittivity and conductivity.

The basic units of the cgs system are cm (centimeter), g (gram), and s (second). But, the fourth unit depends on whether it is based on the cgs electrostatic system of units (e.s.u.) or based on the cgs electromagnetic system of units (e.m.u.).

For the e.s.u. system, the permittivity ε_0 in free space is assigned to be 1 (unrationalized dielectric constant), the fourth basic unit is statcoulomb, and each unit is designated by prefixing the syllable *stat* to the name of the corresponding unit.

For the e.m.u. system, the permeability μ_0 in free space is assigned to be 1 (unrationalized magnetic constant), the fourth basic unit is abampere, and most units are designated by prefixing the syllable *ab* to the name of the corresponding units, [12].

For further details on the cgs-system see `https://en.wikipedia.org/wiki/Centimetregramsecond_system_of_units`. In the following, κ corresponds to the relative permittivity, which is calculated as

$$\kappa = \frac{\varepsilon}{\varepsilon_0},$$

where ε is the permittivity and ε_0 is the vacuum permittivity. A typical value for κ for water at 20 °C is ≈ 80. κ is a dimensionless, substance-specific constant. Minerals like silicon dioxide (quartz) or Mica muscovite ($KAl[Si_3O_{10}(OH)_2]$) have relative permittivities of 3.79 to 4.50 and 5 to 8, [6].

The electrical conductivity σ or specific conductance (which is the reciprocal of the specific electrical resistance) is approximately 1×10^{-14} to $2 \times 10^{-14}\,\Omega^{-1}\,\mathrm{m}^{-1}$ for land. As a rule of thumb, with increasing height above the ground the electrical conductivity of air increases. As an approximation, the average total conductivity σ of air in $1 \times 10^{-14}\,\Omega^{-1}\,\mathrm{m}^{-1}$ up to an altitude of 35 km can be described by the following formula:

$$\sigma = \frac{1}{0.354 \cdot e^{(-4.52 \cdot h)} + 0.125 \cdot e^{(-0.295 \cdot h)} + 0.0225 \cdot e^{(-0.11 \cdot h)}},$$

where h is the altitude in km, [18]. For $h = 3\,\mathrm{km}$, e.g., there results $1.48 \times 10^{-13}\,\Omega^{-1}\,\mathrm{m}^{-1}$.
σ in e.s.u. is converted into SI units by $1\,\mathrm{S\,m^{-1}} = 1 \times 10^{-7} \times c^2$ statmoh cm^{-1}, where mho is a historical unit and is the reciprocal of one ohm. c is the speed of light, $3 \times 10^8\,\mathrm{m\,s^{-1}}$.

1. Introduction

The problem of ground-wave propagation over a homogeneous smooth earth was first solved by SOMMERFELD, [29, 30] and a comprehensive statement of his theory in its most general form has been given by NORTON, [26, 27]. This treatment is, however, based on the assumption that the earth is flat, and it is thus limited in practice to relatively short distances. The rigid solution for a spherical earth was first given by WATSON, [38], though the explicit form of his analysis referred to a perfectly conducting earth. The essential nature of the solution for an imperfectly conducting earth was shown by T. L. ECKERSLEY, [9], and detailed treatments of this generalized problem were given independently by ECKERSLEY and MILLINGTON, [11], VAN DER POL and BREMMER, [34, 33, 32], and VVEDENSKY, [35, 36, 37].

In all the above work, the earth was assumed to be homogeneous, i.e., to be of constant conductivity and permittivity throughout. This is, however, a condition that is far from being realized in practice, and as the solution depends intimately upon the earth constants, it is obvious that this assumption is not justified. Changes in the earth constants from place to place were very irregular and ill-defined, then probably it would be best to assume average values of the conductivity and of the permittivity determined by field-strength measurements. When, however, there are well-defined sections of differing earth constants, as in a transmission path that is partly over land and partly over sea, or which passes from arable land to desert or rock, it becomes necessary to study the propagation over each section and the nature of the transitions across the boundaries.

Unfortunately, the analysis, already very complex, becomes intractable when the condition of homogeneity is waived, and so far no complete solution of the problem has been given. In order to make it feasible at all, the problem must still be idealized by assuming symmetry about a vertical dipole transmitter, i.e., by considering that the various sections are annuli about a vertical axis through the dipole, though this assumption may be far from true, for instance, for a wave passing over a small island in the middle of the sea, where the field-strength beyond the island may be modified by energy flowing round it. It is interesting to note that the recent work of GRUNBERG, [17], and FEINBERG, [15, 16], has been directed towards this aspect of the problem, though it does not appear to be in a form that is applicable to the problem as a whole or amenable to numerical computation.

In view of these difficulties, resort has been made in the past to empirical solutions based on the known solution for the homogeneous earth, and perhaps the best known is the method published as long ago as 1930 by P. P. ECKERSLEY, [8]. Apart

from its empirical nature, this method is open to the objection that it does not satisfy the reciprocity condition, and all such methods are subject to the criticism that a seemingly intelligent guess may be erroneous just because it is a guess and not based on a rigid argument. On the other hand, there is an urgent need for the solution of this problem, for it has long been of great importance in medium-wave broadcasting, and interest in it has recently been stimulated with the development of medium and long-wave navigational aids.

It seems, therefore, that the time is ripe for a review of the subject, and in this paper, we shall discuss the general features that one would expect the rigid solution of the problem to possess, and show to what extent an empirical solution to be proposed fulfils them. It should be borne in mind that the results are admittedly speculative and open to revision in the light of future developments in the complete theory, but it is hoped that they will give estimates of field-strength that are of the right order and provide a reliable guide for the communication engineer faced with practical examples of this difficult problem.

We shall adopt the symmetrical conditions already stated, and assume that the transmitter is a vertical dipole. A wave leaving the transmitter will then pass over a series of sections in each of which the conductivity and permittivity are fixed throughout. In practice the nature and extent of the sections may vary with direction, but the propagation conditions along any given radius will be assessed on the assumption that the same conditions hold along all the other radii. We shall then discuss to what extent this procedure is justified in practice.

It should be noted that we retain throughout the condition that the earth is smooth, so that the inhomogeneity under consideration refers only to the earth irregularities in the terrain. The latter become of major importance on very short waves, but are in general of less significance on the longer waves, where the earth constants affect more intimately the ground-wave propagation characteristics, and the ground-wave range can extend far beyond the horizon. No account is taken of the ionosphere, as the consideration of the relative strengths of ground-wave and sky-wave at a given distance is regarded as a separate subject to be studied after the characteristics of the individual waves are known.

2. General Features to be Expected in the Solution

In the neighbourhood of a transmitter, but beyond the range of the induction field, the radiation initially obeys an inverse-distance law of field-strength, but eventually, with increasing distance, the propagation becomes characteristic of the earth constants, assuming that the earth is homogeneous and smooth. This is true whether the earth is considered flat or spherical, but whereas for the flat earth the variation of field-strength ultimately becomes of inverse-distance-squared type, for the spherical earth the law of attenuation with distance becomes of an exponential type with an attenuation coefficient that is dependent upon the earth constants, except in the limiting cases of very long or very short waves. Accompanying the attenuation with distance, there is a characteristic height-gain effect which, especially close to the ground, depends on the earth constants over the whole range of wavelengths.

If now we consider an inhomogeneous earth in which at a certain distance from the transmitter there is a boundary at which the wave crosses from one type of earth to another, and we suppose that the new section carries on to an indefinite distance, it seems reasonable to expect that at a sufficient distance beyond the boundary the propagation must become characteristic of the earth constants of the new section, both as regards the attenuation with distance and the height-gain relation near the ground. In other words, at a sufficient distance beyond the boundary the type of transmission becomes the same as if the new section had extended right back to the transmitter and the radiated power had been modified in an appropriate way. At the boundary itself, some kind of disturbance must be set up to account for the transition of the propagation from one characteristic type to another. Presumably this disturbance must extend in some degree towards the transmitter, so that the wave must be somewhat modified before it reaches the boundary. It seems reasonable to suppose, however, that this effect will be small compared with the modification beyond the boundary and that it can be neglected in an approximate theory.

In suggesting these features as the probable conditions to be obeyed by the solution of the problem, we are admittedly adopting a speculative line of reasoning, but it is useful to form such preconceived ideas in order to see to what extent any actual suggested solution conforms to them. There is, however, one condition that can be established by rigid analysis, to which the true solution must comply, namely the condition of reciprocity, which for our purposes can be stated in the simple form

that the received field-strength is unaltered if the positions of the transmitter and receiver are interchanged. As mentioned above, it is the failure of the method of P. P. Eckersley to satisfy this essential condition that lays it open to criticism, quite apart from its essentially empirical nature, though it is designed to satisfy our preconceived idea with regard to the attenuation with distance well beyond the boundary.

In order to be able to discuss the probable nature of the solution more explicitly, it is necessary to know the salient features of the solution for the homogeneous earth as regards attenuation with distance and the height-gain relation, and it will be useful to summarize them in a form convenient for the purpose.

3. Summary of Ground-Wave Analysis for a Homogeneous Earth

3.1. The Diffraction Region

The original work of WATSON, as generalized by the later workers mentioned above, showed that the solution of the problem of propagation from a vertical dipole over a homogeneous smooth spherical earth can be represented as the sum of an infinite number of terms, each of which contains a factor that is exponentially attenuated with distance. The attenuation coefficients of the successive terms form a set of increasing numbers derived from an eigenvalue equation, and at a sufficient distance from the transmitter the first term becomes predominant.
It is convenient to refer to the region in which the total field can be represented adequately by the first term as the diffraction region, where the field as a function of the distance d may be expressed by

$$\mathfrak{E}_D = \frac{k}{\sqrt{d}} \cdot e^{-\alpha \cdot d} \cdot f(h_T) \cdot f(h_R) \tag{3.1}$$

where $f(h_T)$ and $f(h_R)$ are the height-gain functions for the transmitter and receiver at heights h_T and h_R respectively, defined to be unity for zero height. The value of k depends upon the radiated power, and it is a function of the earth constants and the wavelength. The coefficient α is a function of the wavelength and in general of the earth constants, but at the short-wave and long-wave limits it is simply proportional to $\lambda^{-\frac{1}{3}}$, where λ is the wavelength, i.e. $\alpha \cdot \lambda^{\frac{1}{3}}$ has upper and lower limiting values.[1]
The condition for the predominance of the first term of the diffraction formula is mainly controlled by the difference between the $\alpha \cdot d$ values for the first and second terms. It can be shown that their difference depends somewhat on the value of the earth constants, but that a useful criterion for all wavelengths for the diffraction region, when both the transmitter and receiver are on the ground, is $d > 200 \cdot \lambda^{\frac{1}{3}}$, where d and λ are both measured in kilometres.[2] Where the transmitter and receiver are several wavelengths above the ground, as is usually the case on very

[1] These limits occur where $\sigma^{\frac{1}{2}} \cdot \lambda^{\frac{5}{6}} < 1 \times 10^{-8}$ and $> 1 \times 10^{-6}$, respectively. (σ in e.m.u. and λ in km)

[2] The value $200 \cdot \lambda^{\frac{1}{3}}$ corresponds to an error of about 3 dB in the diffraction formula as given by the first term only. For an error of 1 dB this distance must be increased to about $300 \cdot \lambda^{\frac{1}{3}}$.

high frequencies, this distance must be referred to the horizon as datum.
Throughout the analysis, the earth constants occur combined in a function that we shall call ζ, and that is defined differently according as the polarization is vertical or horizontal, i.e., according as the transmitting aerial is a vertical electric or magnetic dipole. For vertical and for horizontal polarization we have

$$\textit{vertical}: \quad \zeta = \frac{\kappa - \mathrm{i}\, 2\sigma \cdot \lambda c}{\sqrt{\kappa - 1 - \mathrm{i}\, 2\sigma \cdot \lambda c}} \tag{3.2a}$$

$$\textit{horizontal}: \quad \zeta = \frac{1}{\sqrt{\kappa - 1 - \mathrm{i}\, 2\sigma \cdot \lambda c}}\,, \tag{3.2b}$$

where κ is the permittivity of the earth referred to unity for air, and σ is the conductivity in e.m.u. The wavelength λ must here be measured in centimetres if the velocity of light c is reckoned in centimetres per second. The time factor is assumed to be

$$e^{\mathrm{i}\,\omega t}\,.$$

For the case of vertical polarization, the field $\mathfrak{E}_D$ in Equation 3.1 is the vertical component of the electric field, or alternatively the transverse horizontal magnetic field, while for horizontal polarization the roles of the electric and magnetic field must be interchanged throughout. On the medium and long waves, where the problem we are investigating is of special importance, we shall be concerned only with vertical polarization, and shall assume that the polarization is vertical unless otherwise stated. It is, however, interesting to note that the whole of the argument will apply to horizontal polarization provided that we use the appropriate form of ζ and interchange the electric and magnetic fields.

3.2. The Sommerfeld Region

The complete diffraction formula holds at all distances from the transmitter, but as the transmitter is approached, more and more terms of the series have to be included and the formula becomes impracticable for general use. Fortunately, for some distance from the transmitter, the effect of the earth's curvature is small and the field-strength can be adequately represented by the Sommerfeld formula for a flat earth. This region we shall therefore refer to as the Sommerfeld region, and between it and the diffraction region there is a transition region in which it is usual to obtain the field-strength/distance curve by interpolation by eye.
The Sommerfeld analysis, as expressed by Norton, represents the field as the sum of a space wave and a surface wave. The space wave is itself the sum of a direct free-space wave and a wave reflected from the earth, and is zero when both transmitter and receiver are on the ground. On very short waves the space wave very rapidly predominates over the surface wave as either the transmitter or receiver is raised above the ground, but on medium waves the surface wave is

predominant, except at great heights, for vertical polarization.
Both the surface wave and the space wave have to be taken into consideration in studying the height-gain effects, though we shall be mainly concerned with the case where the transmitter and receiver are in effect on the ground. The field-strength is then given by the inverse-distance value corresponding to a perfectly conducting earth, modified by an attenuation factor that is a function of the earth-constant factor ζ, the wavelength and the distance, combined in a quantity known as the numerical distance w given by

$$w = -\mathrm{i} \cdot \frac{\pi d}{\lambda \zeta^2} \,,$$

where d and λ are in the same units.
Denoting the field in the SOMMERFELD region by $\mathfrak{E}_S$, we have

$$\mathfrak{E}_S = \frac{K}{d} \cdot F(w) \,, \tag{3.3}$$

where K is a constant depending on the radiated power, and $F(w)$ is the attenuation factor. The properties of $F(w)$ are discussed in detail by NORTON in terms of the modulus and phase of w. When $|w| < 0.01$, $F(w)$ is effectively unity, but as w is increased, $F(w)$ decreases, and when $|w| > 20$, $F(w)$ approximates to $-\frac{1}{2}w$. We thus have

$$\mathfrak{E}_S = \frac{K}{2\pi \cdot d^2} \cdot \lambda \cdot |\zeta|^2 \tag{3.4}$$

if

$$d > \frac{20}{\pi} \cdot \lambda \cdot |\zeta|^2 \,,$$

where we need to consider only the modulus of the field.
Under these conditions the field obeys an inverse-distance-squared law, and this law comes into operation where the field has dropped about 32 dB below the inverse-distance value. We have, however, to remember that in practice the curvature of the earth may come into play before this condition is reached, depending on the relative sizes of the critical distances $200 \cdot \lambda^{\frac{1}{3}}$ and $\frac{20}{\pi} \cdot \lambda \cdot |\zeta|^2$.
With increasing wavelength, the first of these distances becomes smaller than the second, and at the long-wave limit the transition is directed from the inverse-distance law to the diffraction region. On the other hand, at the short-wave limit the inverse-square law takes charge before the curvature of the earth appreciably modifies the flat-earth value of the field-strength. The wavelength at which the field begins effectively to go straight over from the inverse-distance law to the diffraction region depends on the assumed earth constants. For transmission over sea, for which we may take $\kappa = 80$ and $\sigma = 4 \times 10^{-11}$ e.m.u., this type of transition occurs for $\lambda > 30$ m, while for transmission over land, with $\kappa = 5$ and $\sigma = 1 \times 10^{-13}$ e.m.u., it occurs for $\lambda > 2000$ m.

3.3. Height-Gain Relations for Small Heights

In the diffraction formula, each term of the series can be shown to be the resultant of an incident and a reflected wave. In the immediate neighbourhood of the ground these waves behave as though they were parts of infinite plane waves with reflection taking place according to the FRESNEL formula.[3] It was shown by WEYL, [39], that the SOMMERFELD analysis for the flat earth is equivalent to an infinite set of plane waves similarly reflected. In both cases the angles of incidence, or rather their direction cosines, are complex, implying amplitude variations over the wave front. If we denote by n the sine of the angle of elevation of the incident wave, the reflection factor is given by

$$\rho = \frac{(\zeta_n - 1)}{(\zeta_n + 1)}\,. \tag{3.5}$$

Strictly speaking, the denominator of ζ in Equation 3.2a and Equation 3.2b should here be

$$\sqrt{\kappa - 1 + n^2 - \mathrm{i}\,2\sigma \cdot \lambda c}\,.$$

If we retain only the variation of the fields of the incident and reflected waves with height, we may write the relative values as

$$e^{\mathrm{i}\left(\frac{2\pi}{\lambda}\cdot nh\right)}$$

and

$$\rho \cdot e^{-\mathrm{i}\left(\frac{2\pi}{\lambda}\cdot nh\right)}\,,$$

and for small heights the combined field is

$$(1+\rho) + \frac{(1-\rho)\,\mathrm{i}\,2\pi \cdot nh}{\lambda}\,,$$

compared with the field

$$(1+\rho)$$

on the ground. The initial form of the general height-gain factor is therefore

$$f(h) = 1 + \frac{(1-\rho)}{(1+\rho)} \cdot \mathrm{i}\,\frac{2\pi \cdot nh}{\lambda}\,.$$

From Equation 3.5 we have

$$\frac{(1-\rho)}{(1+\rho)} = \frac{1}{\zeta_n}$$

[3] A particular class of diffraction effects is the "FRESNEL diffraction", which refers usually to, although not exclusively, the phenomena observed close to a two-dimensional object illuminated by plane parallel incident light, [13]. See also "What is Fraunhofer and Fresnel Diffraction?" at `https://www.gophotonics.com/community/what-is-fraunhofer-and-fresnel-diffraction`

so that

$$f(h) = 1 + \mathrm{i}\,\frac{2\pi \cdot h}{\lambda \cdot \zeta}\,. \tag{3.6}$$

This result is of particular interest because it is practically independent of the value of n, and is therefore true for any system of such incident and reflected waves. ECKERSLEY and MILLINGTON showed directly from the diffraction analysis that this result for the initial height-gain holds for a single term of the formula, and it was confirmed by VAN DER POL and BREMMER. As it is true for the sum of any number of such terms, it must also hold in the transition and SOMMERFELD regions so long as the angle of elevation is small, and it is not difficult to show from NORTON's expressions for the space and surface waves in the inverse-distance-squared region that their sum does initially obey this relation.
For vertical polarization we can see from Equation 3.2a that the phase angle of ζ lies between 0 and $-\pi/4$, and hence the modulus of $f(h)$ initially decreases. This phenomenon was pointed out by VVEDENSKY, who derived it by actual computation of the height-gain factor for a particular case, but we can see that fundamentally it arises as a property of FRESNEL reflection.
At the short-wave limit, where from Equation 3.2a ζ is wholly real, this initial drop disappears, while at the long-wave limit, where the phase angle of ζ is $-\pi/4$, the initial drop does not develop, partly because we cannot in practice make $\frac{2\pi \cdot h}{\lambda \cdot \zeta}$ comparable with unity, and partly because the assumption that the wave is plane would not hold over a sufficient range of height. The height-gain factor at this limit is in effect constant and equal to unity. On intermediate wavelengths, however, the initial drop can reach its maximum of about 3 dB.

3.4. Height-Gain in the Diffraction Region

It was first pointed out by T. L. ECKERSLEY that the height-gain function for each term of the diffraction formula is intimately associated with a certain concentric fictitious sphere of radius somewhat greater than that of the earth. Above this sphere the height-gain function rapidly assumes an exponential form, and apart from a slowly varying factor, the height-gain due to an increase in the height is given by

$$e^{\alpha \cdot d'}\,,$$

where d' is the increase in the horizon distance to the fictitious sphere, and α is the same coefficient as in the attenuation factor in Equation 3.1. We must remember that above the fictitious sphere an increase in height is equivalent to increasing by the corresponding distance d' the limit of the diffraction region.
From this result ECKERSLEY made the observation that above the fictitious sphere an increase in height was equivalent to remaining the same height and moving back towards the transmitter by the corresponding distance d', and alternatively that along a tangent to the fictitious sphere, the wave behaves like an unattenuated

progressive wave.
The height of the fictitious sphere above the earth is proportional to $\lambda^{\frac{2}{3}}$. It depends somewhat on the value of the earth-constant function ζ, but the dependence is small, and to a reasonable approximation the height, say h_0, may be taken for all conditions to be

$$h_0 = 50 \cdot \lambda^{\frac{2}{3}} , \tag{3.7}$$

where h_0 and λ are both measured in metres.
On very short waves, h_0 may be small compared with the transmitter and receiver heights, but on long waves it plays a major part in the practical application of the theory in deciding whether there will be any appreciable height-gains.

3.5. Height-Gain at the Short-Wave Limit

At the short-wave limit the value of k in Equation 3.1 is proportional to ζ^2, and the attenuation coefficient α has a limiting form proportional to $\lambda^{-\frac{1}{3}}$ and is independent of the earth constants. Under these conditions, the form of the height-gain function given in Equation 3.6 holds at heights where $\frac{2\pi \cdot h}{\lambda \cdot \zeta} \gg 1$, so that $f(h) \simeq \mathsf{i}\,\frac{2\pi \cdot h}{\lambda \cdot \zeta}$. If we define a height h_1 as that at which this approximation to $f(h)$ becomes valid, we see that for heights greater than h_1, where Equation 3.6 still holds, $f(h)$ becomes proportional to ζ^{-1}.
It can be shown that at greater heights above the fictitious sphere, where $f(h)$ becomes of exponential form, $f(h)$ is still proportional to ζ^{-1}; in fact at all heights greater than h_1, even where Equation 3.6 no longer holds, this relationship is true. If both transmitter and receiver are raised to heights greater than h_1, the combined height-gain effect $f(h_T) \cdot f(h_R)$ in Equation 3.1 is proportional to ζ^{-2}, and as $k \propto \zeta^2$, it follows that the field-strength is independent of ζ, i.e., of the earth constants, since α is similarly independent under our limiting conditions.
This argument is not confined to the diffraction region, for where the inverse-distance-squared law holds, the field $\mathfrak{E}_S$ in Equation 3.4, with $h_T = h_R = 0$, is proportional to ζ^2. At the short-wave limit this law comes into operation in the Sommerfeld region, while the initial form of the height-gain function in Equation 3.6 holds at least up to the height h_1.
We thus reach the very general conclusion that at the short-wave limit the field-strength becomes independent of the earth constants when both the transmitter and the receiver are above the height h_1, and that this result holds even beyond the horizon in the diffraction region. It is important to notice that at the short-wave limit, where $\kappa \gg 2\sigma \cdot \lambda c$ in Equation 3.2a, and the earth behaves as a pure dielectric, ζ is still different for, say, land and sea, and the ground-to-ground field-strength (i.e. with $h_T = h_R = 0$) is different. The initial height-gain is, however, different, and above h_1, for both transmitter and receiver, these two factors compensate, and further increases in height give the same increase in field-strength whatever the value of ζ.

4. Solution at the Short-Wave Limit for the Inhomogeneous Earth

Let us now consider a transmission path of length d made up of two sections d_1 and d_2 with earth-constant functions ζ_1 and ζ_2, shown in Figure 4.1, drawn flat for convenience, with the transmitter and receiver both on the ground, at T and R. Confining our attention to the short-wave limit, let us imagine T and R both raised to a height h_1 at T' and R' respectively, where we choose the greater of the values for h_1 corresponding to ζ_1 and ζ_2. Then from the above argument, the received field-strength at R' from the transmitter at T' is independent of the earth constants of the ground below, and hence of the fact that it consists of two sections of different ζ.

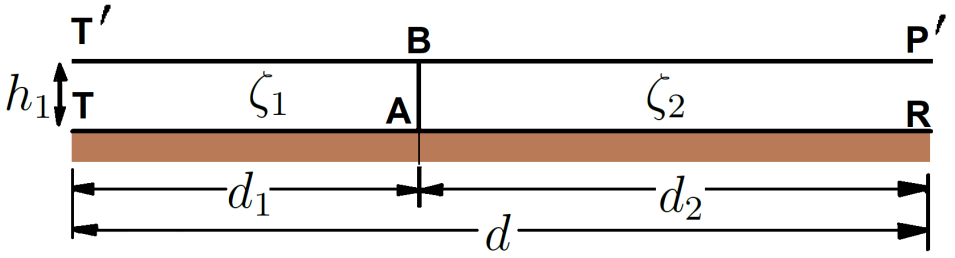

Figure 4.1.: Single boundary for the short-wave limit, showing the height h_1

At the point B at the height h_1 above the boundary at A there is no discontinuity of field, as the attenuation with distance is the same for both sections, and above B the height-gain becomes independent of the earth constants. If $\mathfrak{E}_1(d)$ is the ground-to-ground field-strength on the assumption that the earth is homogeneous over the whole route and of type ζ_1, and $f_1(h)$ is the corresponding height-gain function, the field at R' due to T', say $\mathfrak{E}_{T'R'}$, can be represented by

$$\mathfrak{E}_{T'R'} = \mathfrak{E}_1(d) \cdot [f_1(h_1)]^2 \ . \tag{4.1}$$

Alternatively, assuming that the ground is all of type ζ_2, we may write

$$\mathfrak{E}_{T'R'} = \mathfrak{E}_2(d) \cdot [f_2(h_1)]^2 \ . \tag{4.2}$$

We can see that these two values are equivalent, for we have shown that

$$\frac{\mathfrak{E}_1(d)}{\mathfrak{E}_2(d)} = \frac{\zeta_1^2}{\zeta_2^2} \tag{4.3}$$

and that for $h \geq h_1$,

$$\frac{f_1(h)}{f_2(h)} = \frac{\zeta_2}{\zeta_1}\,. \tag{4.4}$$

Now if the boundary at A is sufficiently far from both T and R, the height-gain at T will actually be of type $f_1(h)$ and at R it be of type $f_2(h)$. Thus, if we move the transmitter and receiver back from T' and R' to T and R, the actual ground-to-ground field-strength $\mathfrak{E}(d)$ over the composite path may be found from

$$\mathfrak{E}(d) = \frac{\mathfrak{E}_{T'R'}}{f_1(h_1) \cdot f_2(h_1)}$$

and from Equation 4.1 and Equation 4.2 this may be written in the alternative forms

$$\mathfrak{E}(d) = \mathfrak{E}_1(d) \cdot \frac{f_1(h_1)}{f_2(h_1)}$$

and

$$\mathfrak{E}(d) = \mathfrak{E}_2(d) \cdot \frac{f_2(h_1)}{f_1(h_1)}\,.$$

That these are equivalent again follows from Equation 4.3 and Equation 4.4, but by multiplying them together we obtain the simple relation

$$\mathfrak{E}(d) = \sqrt{\mathfrak{E}_1(d) \cdot \mathfrak{E}_2(d)}\,, \tag{4.5}$$

i.e., the ground-to-ground field-strength is the geometric mean of the values obtained on the assumption that the ground is first all of type ζ_1 and then all of type ζ_2, where again ζ_1 and ζ_2 are combined functions of the earth constants. This result has recently been given by T. L. Eckersley, [10] specifically for the case where the boundary is in the diffraction region of the transmitter.
This method can obviously be extended to the case of n sections, provided that the first and last boundaries are sufficiently far from T and R respectively, the result being of the form

$$\mathfrak{E}(d) = \sqrt{\mathfrak{E}_1(d) \cdot \mathfrak{E}_n(d)} \tag{4.6}$$

and depending only on the nature of the first and last sections.
Returning to the single boundary at A in Figure 4.1, if we assume, as suggested earlier, that the nature of the propagation is unaffected before the boundary is reached, the height-gain from A to B will be of the type $f_1(h)$. The propagation beyond A towards R cannot become immediately characteristic of ζ_2 with a height-gain function $f_2(h)$. There must be a certain distance beyond A over which the

field-strength on the ground and the height-gain function adjust themselves in such a way that the height-gain eventually becomes of type $f_2(h)$, and that at the height h_1 the field beyond B is attenuated with distance as though there were no boundary to consider.
This adjustment may be attributed to some kind of disturbance function. We may say that above the height h_1 the wave to the right of the boundary is the same as if it had proceeded from a transmitter of modified power at T over ground wholly of type ζ_2, the modification in power being required to fit the wave at all heights above B on to the actual wave to the left of the boundary that has come from the actual transmitter at T over ground of type ζ_1. If at the boundary we attempt to carry these waves below h_1 to the ground, the different height-gain functions on the two sides of the boundary would mean that the field on the ground would change discontinuously at A, and we may regard the disturbance function as removing this discontinuity. We are assuming tentatively that the function does not extend back towards the transmitter and that it dies out at the height h_1. Of its extent beyond the boundary we can say little at this stage of the argument, but apart from this uncertainty we have obtained a complete solution in this limited case.

5. The General Case in the Diffraction Region

At the long-wave limit, where in the diffraction region the attenuation coefficient again becomes independent of the earth constants, and where in the SOMMERFELD region the inverse distance law holds, the earth behaves as if it were a perfect conductor, and inhomogeneities in the earth constants do not affect the ground-wave propagation. It is at the intermediate wavelengths, where in the diffraction region the attenuation coefficient α is a function of the earth-constant function ζ, that the greatest difficulties in the theory arise. It is possible, for instance, on some wavelengths for α to be twice as great for propagation over land as for over sea, causing extreme differences in attenuation with distance. This is offset by the difference in the initial height-gain, and at great heights by the fact that above the fictitious sphere the height-gain function becomes of the type $e^{\alpha \cdot d'}$. We see, however, that at a boundary we can no longer fit the two waves together at all heights above a certain height, both on account of the different height-gain relations and because of the different characteristic attenuation with distance on the two sides of the boundary.
In order to obtain an idea of the probable form of the solution in this general case, we shall restrict our argument and consider initially a single boundary in the diffraction region with respect to both the transmitting point T and the receiving point R as in Figure 5.1, where both T and R are on the ground. Instead of considering the height h_1 as in Figure 4.1, we use the height of the fictitious sphere h_0 given by Equation 3.7, treating it as independent of the earth constants, i.e., of the functions ζ_1 and ζ_2 for the two sections, and we take B as the point on the fictitious sphere above the boundary A.

Assuming that from the transmitter to A the field at all heights is characteristic of the type ζ_1, it seems reasonable to suppose from the argument given in section 3.4 that the field is unmodified beyond A at all points above the tangent to the fictitious sphere at B. Beyond A there will be a disturbance function, and we will suppose that it is necessary to go as far as A' before the attenuation with distance has become characteristic of type ζ_2, and height-gain function has become of type $f_2(h)$ for all points below B' on the fictitious sphere. We then assume that beyond A' the height-gain function is of type $f_2(h)$ for all points below the tangent to the fictitious sphere at B'.
In the case of the homogeneous earth we can construct from results of section 3.4

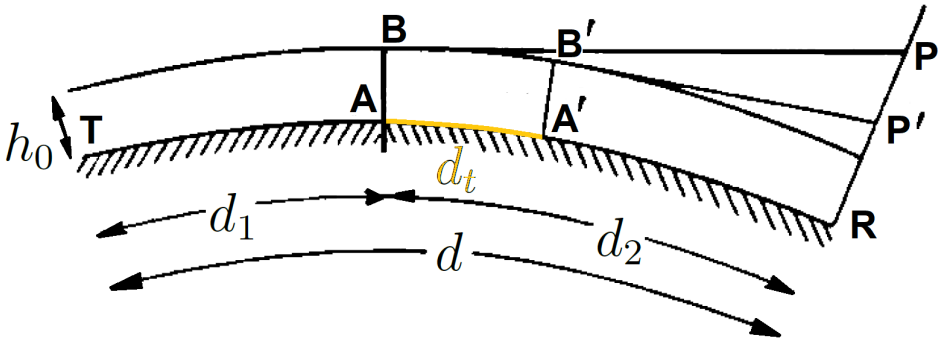

Figure 5.1.: Single boundary in the diffraction region for the general case, showing the height h_0

the physical picture in the diffraction region of a waveguide of width h_0, through the upper side of which energy flows out tangentially to supply the field in the space above the guide. The significance of the fictitious sphere as the upper boundary of a leaky waveguide was first pointed out by Booker, [2]
In the present argument, the fact that h_0 is approximately independent of the earth constants is used to justify a similar physical picture, and the redistribution of energy taking place in the guide between A and A' affects the height-gain relationship at all heights between the tangents at B and B'. If these tangents cut the vertical through R at P and P', we can say that above P the height-gain is of type $f_1(h)$ and that below P' it is of type $f_2(h)$, while between P' and P, as we go up from R, the height-gain is changing from type $f_2(h)$ to type $f_1(h)$.
At the short-wave limit the two functions $f_1(h)$ and $f_2(h)$ are of the same type at all points above h_0, since it can be shown that h_0 is there much greater than h_1, and the above physical picture becomes consistent with the argument of chapter 4, though it is here formally restricted to the diffraction region. According to this picture the field-strength at P can be calculated from the ground-to-ground field-strength $\mathfrak{E}_1(d)$ for an earth all of type ζ_1 with a height-gain factor of type $f_1(h)$ for the height RP. From the analysis in section 3.4 it follows that this height-gain is given approximately by

$$f_1(h_0) \cdot e^{\alpha_1 \cdot d_2} .$$

Thus, we have

$$\text{Field at } P = \mathfrak{E}_1(d) \cdot f_1(h_0) \cdot e^{\alpha_1 \cdot d_2} .$$

For the reduction in field-strength in going from P to P', corresponding to the transition distance AA', say d_t, we shall assume that it is given by an exponential with an average coefficient $\frac{1}{2} \cdot (\alpha_1 + \alpha_2)$, so that we have

$$\text{Field at } P' = \mathfrak{E}_1(d) \cdot f_1(h_0) \cdot e^{\left\{\alpha_1 \cdot d_2 - \frac{1}{2} \cdot (\alpha_1 + \alpha_2) \cdot d_t\right\}} .$$

Finally, to obtain the field at R, we must divide by the height-gain from R to P'

which we have seen is of type $f_2(h)$, and which to our approximation is given by

$$f_2(h_0) \cdot e^{\{\alpha_2 \cdot (d_2 - d_t)\}} ,$$

so that the actual ground-to-ground field-strength is given by

$$\begin{aligned} \mathfrak{E}(d) &= \mathfrak{E}_1(d) \cdot \frac{f_1(h_0)}{f_2(h_0)} \cdot e^{\left\{\alpha_1 d_2 - \frac{1}{2} \cdot (\alpha_1 + \alpha_2) \cdot d_t - \alpha_2 \cdot (d_2 - d_t)\right\}} \\ &= \mathfrak{E}_1(d) \cdot \frac{f_1(h_0)}{f_2(h_0)} \cdot e^{\left\{(\alpha_1 - \alpha_2) \cdot d_2 - \frac{1}{2} \cdot (\alpha_1 - \alpha_2) \cdot d_t\right\}} . \end{aligned} \tag{5.1}$$

This value does not essentially obey the reciprocity condition and if we interchange T and R and assume that the transition distance has the same value d_t, we have as an alternative value

$$\mathfrak{E}(d) = \mathfrak{E}_2(d) \cdot \frac{f_2(h_0)}{f_1(h_0)} \cdot e^{\left\{(\alpha_2 - \alpha_1) \cdot d_1 - \frac{1}{2} \cdot (\alpha_2 - \alpha_1) \cdot d_t\right\}} . \tag{5.2}$$

It is an argument in favour of the approximate correctness of our theory that these values are found by calculation to be nearly equal even in extreme cases. To make the result strictly reciprocal and to obtain agreement at the short-wave limit with the value in Equation 4.5, we take the geometric mean, giving

$$\mathfrak{E}(d) = \sqrt{\mathfrak{E}_1(d) \cdot \mathfrak{E}_2(d)} \cdot e^{\left\{\frac{1}{2} \cdot (\alpha_1 - \alpha_2) \cdot (d_2 - d_1)\right\}} . \tag{5.3}$$

As in section 3.4, we have omitted a slowly varying factor in the height-gain function, but it can be shown that the small corrections its inclusion would make to Equation 5.1 and Equation 5.2 cancel out when the geometric mean is taken, leaving Equation 5.3 unaltered. Further, in the transition region $P'P$ we could assume for the average attenuation coefficient the more general form

$$l \cdot \alpha_1 + m \cdot \alpha_2 ,$$

where $l + m$ must be equal to unity to give the correct form at the limit when α_1 and α_2 are equal, and we should again find that it would not affect the value of the geometric mean.
We are therefore encouraged to believe that our admittedly approximate physical picture is reasonably good, and that it leads to a value of $\mathfrak{E}(d)$ that is of the right order. It is only a matter of detail to generalize the method to the case of n sections, subject to the conditions laid down for the diffraction region, and the result is found to be

$$\mathfrak{E}(d) = \sqrt{\mathfrak{E}_1(d) \cdot \mathfrak{E}_n(d)} \cdot e^{\left\{\frac{1}{2} \cdot (\alpha_1 + \alpha_n) \cdot d - \alpha_1 d_1 - \alpha_2 d_2 - \cdots - \alpha_n d_n\right\}} , \tag{5.4}$$

which reduces to Equation 5.3 when $n = 2$, and to Equation 4.6 when the α's are all equal.

The special case of a single boundary given in Equation 5.3 is contained in ECKERSLEY's paper, but it is there deduced on the assumption that the height-gain functions are approximately of the same form, and that the waves on the two sides of the boundary can still be fitted together, except near the ground, as in the argument for the short-wave limit. For this reason, he states that the method is definitely unsuitable for longer waves. The derivation given above waives the condition that the height-gain functions should be approximately of the same form, and is based on the concept of the energy flow above the fictitious sphere with a transition region extending above the fictitious sphere between the two tangents defined. It thus removes the restriction imposed by ECKERSLEY, and justifies his tentative use of the method for longer waves.
We may note from Equation 5.3 that for the case of the single boundary the correction factor,

$$e^{\left\{\frac{1}{2}\cdot(\alpha_1-\alpha_2)\cdot(d_2-d_1)\right\}},$$

to the geometric mean value

$$\sqrt{\mathfrak{E}_1(d)\cdot\mathfrak{E}_2(d)},$$

is unity not only at the short-wave limit where $\alpha_1 = \alpha_2$, but also when $d_1 = d_2$, whatever the wavelength, i.e., when the boundary is mid-way between T and R. In his application of the method to the medium-wave broadcast band, ECKERSLEY has taken the simple geometric mean, and the general agreement of his calculated values with the measured values for a number of continental stations received in England implies that for the paths considered the correction factor is of the order of unity.
If in Equation 5.4 we give $\mathfrak{E}_1(d)$ and $\mathfrak{E}_n(d)$ their values

$$\mathfrak{E}_1(d) = \frac{k_1}{\sqrt{d}}\cdot e^{-\alpha_1\cdot d}$$

and

$$\mathfrak{E}_n(d) = \frac{k_n}{\sqrt{d}}\cdot e^{-\alpha_n\cdot d},$$

we have

$$\mathfrak{E}(d) = \sqrt{\frac{k_1\cdot k_n}{d}}\cdot e^{\{-\alpha_1\cdot d_1-\alpha_2 d_2-\cdots-\alpha_n d_n\}} \tag{5.5}$$

and as the distance d_n of the receiver from the last boundary is increased, the field as a function of d_n becomes proportional to $e^{-\alpha_n\cdot d_n}$. Our solution thus conforms to our preconceived idea that at a sufficient distance from the last boundary the attenuation with distance becomes characteristic of the type of earth over which the wave is travelling.
Returning to the situation above the fictitious sphere in the case of the single

boundary represented in Figure 5.1, let us consider the change in field-strength with distance at a given height shown by the blue chain line in Figure 5.2.

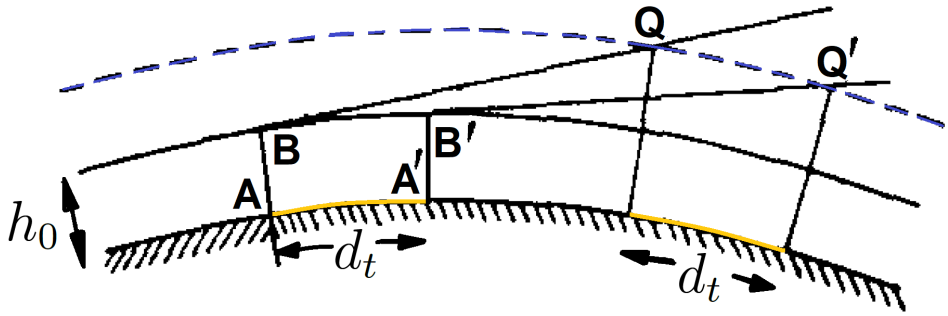

Figure 5.2.: Single boundary in the diffraction region, showing the transition distance at a height above h_0

If this line cuts the tangents to the fictitious sphere shown in Figure 5.2 at Q and Q', it will be seen that the movement from Q to Q' corresponds to a distance along the ground equal to the transition distance $AA' = d_t$, whatever the height above the earth. It follows that the field-strength/distance curves at various heights above the fictitious sphere in the diffraction region become of the same shape, being laterally displaced with respect to one another by distances given by the differences in the horizon distance to the fictitious sphere. Each curve will have a transition region of length d_t, over which the slope changes from the slope characteristic of one type of earth to that of the other type beyond the boundary.
In Figure 5.3 a diagrammatic representation of a set of field-strength/distance curves for various heights above the fictitious sphere is shown for the case of a wave crossing a boundary to a section of higher conductivity, e.g., from land to sea. The transition region in each curve is shown as a broken line. The horizontal lines CD and $C'D'$ at the boundaries of the transition region correspond to the unattenuated propagation along the tangents at B and B' in Figure 5.1 and Figure 5.2. Above CD and below $C'D'$ the curves have the characteristic slopes for the sections before and beyond the boundary respectively.
This diagram shows clearly how the difficulty of fitting together the waves with their different height-gain relations at all heights is overcome by means of the transition region defined by the tangents at B and B'. We can see that the greater height-gain at the vertical line U in Figure 5.3, as compared with that shown at V, is offset by the steeper slope of the curves above CD as compared with the slope below $C'D'$, so that the curves for the different heights have the same shape as one another throughout their length, and may be obtained from one another by a lateral displacement.
Although the curves in Figure 5.3 are meant only to be diagrammatic, they have been made to correspond to the case of a transmission on a wavelength of 100 m

across a boundary from land of medium conductivity ($\sigma = 1 \times 10^{-13}$ e.m.u.) to sea ($\sigma = 4 \times 10^{-11}$ e.m.u.). The curves above CD are obtained by applying the land height-gain to the overland ground curve, while for the curves below $C'D'$ the sea height-gain has been added to the ground curve given by Equation 5.3. For a reason to be given later, the transition distance d_t has been taken as the critical distance $200 \cdot \lambda^{\frac{1}{3}}$ defined above in connection with the limit of the diffraction region. It is interesting to note that the portion of a curve above CD can be joined to the portion below $C'D'$ with a smooth curve in the transition region, suggesting that Equation 5.3 does give a reasonable value for the ground-to-ground curve at a sufficient distance beyond the boundary in the diffraction region.

The above picture should be modified slightly to include the slowly varying factor in the height-gain function and the $\frac{1}{\sqrt{d}}$ term in the diffraction formula. Moreover, the exponential form of the height-gain function does not come fully into operation until one is well above the fictitious sphere, but by concentrating on the major factors we have obtained an assessment of the situation above the fictitious sphere that would seem to be correct in its essential features.

The problem remains of investigating the nature of the transition of the field-strength on the ground from the one type of transmission to the other, e.g., from a curve with a slope characteristic of overland transmission to one characteristic of oversea transmission. If we take again a wavelength of 100 m, for which the fictitious sphere is at a height, from Equation 3.7, of about 1100 m, the oversea height-gain presents an initial drop and does not become positive until a height of nearly 2000 m, whereas the overland height-gain becomes positive at about 150 m and is 25 dB at 2000 m. On the assumption that the field is unmodified at all heights until the boundary is reached, we can construct Figure 5.4, in which the field-strength/distance curves are shown to the left of the boundary for various heights up to 5000 m. Beyond the transition region, the ground curve given by Equation 5.3 is shown on the right, and the corresponding curves for the heights shown on the left actually lie below it by amounts ranging up to 3 dB, decreasing to zero at 2000 m. A tentative curve for the transition at 2000 m is shown, and it is obvious that below this height the form of the transition must change very rapidly as we pass through and below the fictitious sphere. At the height of 5000 m, shown in Figure 5.4, the curve is approaching the type in Figure 5.3 in the region where the curves have the same shape and are merely laterally displaced.

A transition curve joining up the two parts of the ground curve is also shown in Figure 5.4. This is admittedly speculative, and in order to explain how it has been derived, we must now turn to an empirical solution, based on the observation, as we shall see, that the method proposed by P. P. Eckersley, referred to above, is in fact related to the solution we have derived.

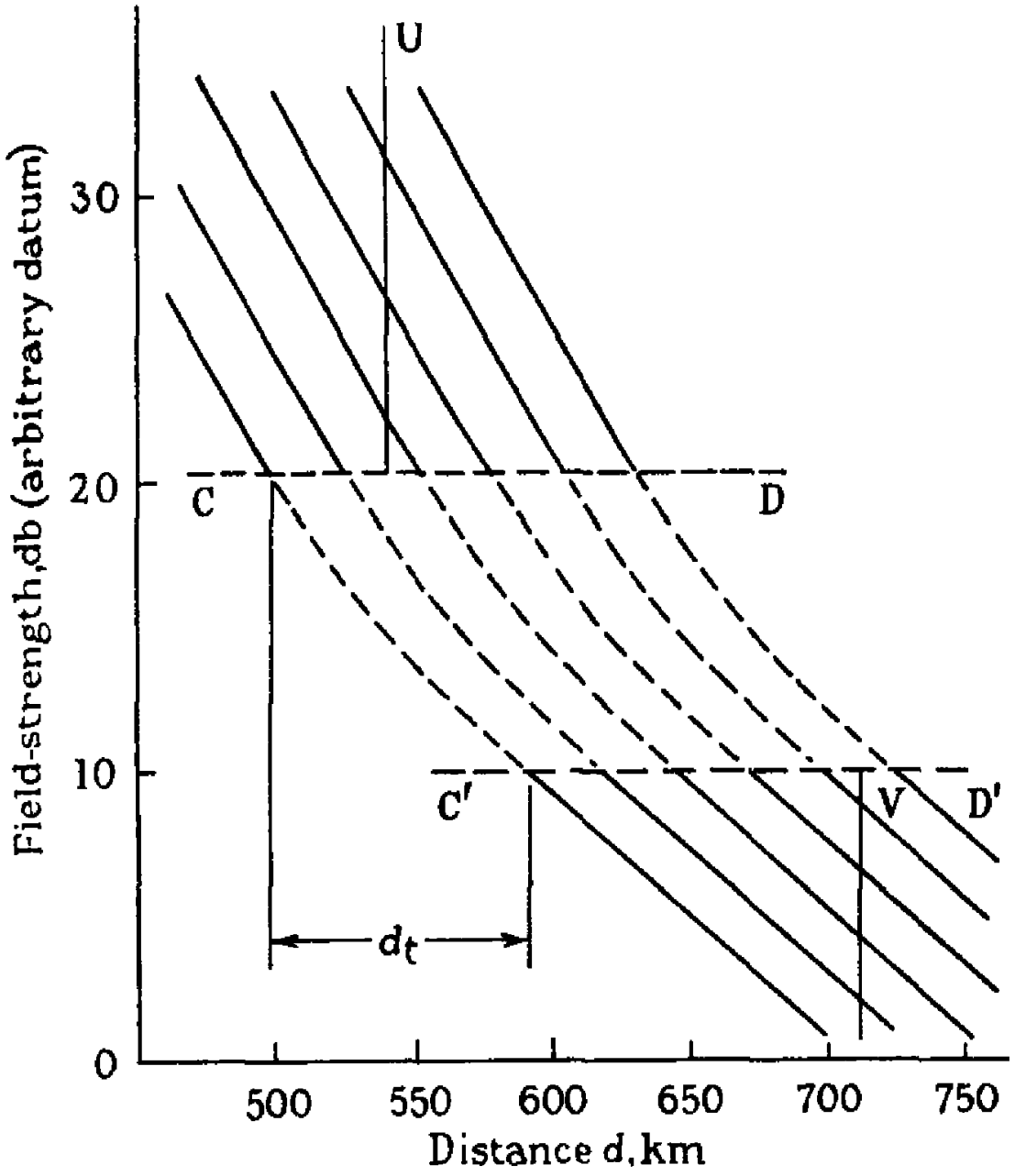

Figure 5.3.: Field-strength/distance curves at heights well above h_0:
$\lambda = 100\,\mathrm{m}$
$\sigma = 1 \times 10^{-13}$ e.m.u. (land)
$\sigma = 4 \times 10^{-11}$ e.m.u. (sea)

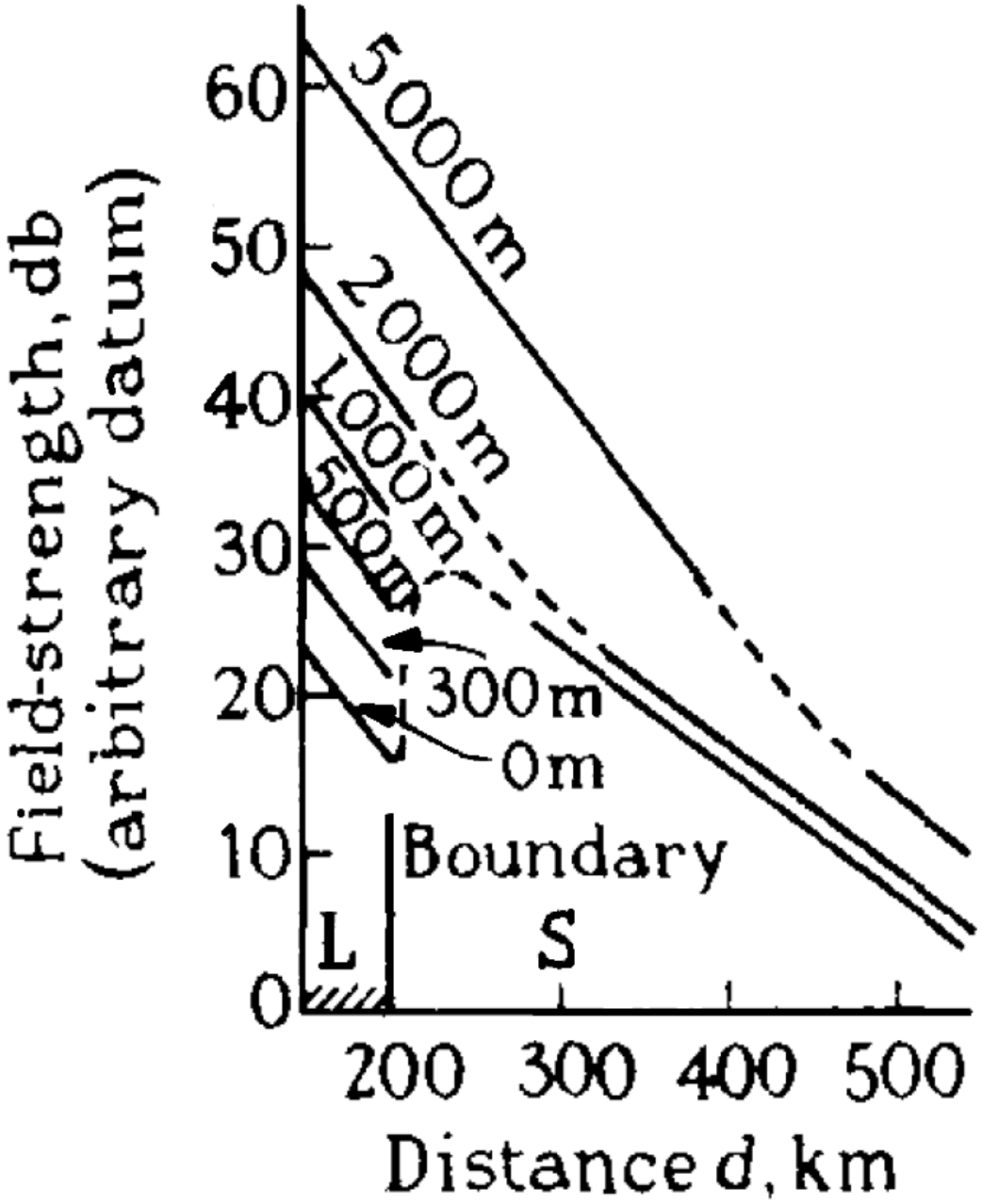

Figure 5.4.: Field-strength/distance curves suggesting the transition from small to great heights:
$\lambda = 100\,\mathrm{m}$
$\sigma = 1 \times 10^{-13}$ e.m.u. (land)
$\sigma = 4 \times 10^{-11}$ e.m.u. (sea)

6. An Empirical Solution in the Neighbourhood of a Boundary

P. P. Eckersley's method was to plot the field-strength/distance curves for propagation over various types of earth on a logarithmic scale of field-strength, i.e., a linear scale in decibels, and then to build up a composite curve by moving the sections of the appropriate curves up or down until they fitted together at the various boundaries. Suppose, for instance, that we have a transmission over a distance d_1 of sea followed by a section d_2 of land. We start along the sea curve, as in Figure 6.1, as far as the point A at a distance d_1 from the transmitter. If we had used the land curve instead, we should have come to a point B at the same distance, and by continuing a further distance d_2 along this curve, should have arrived at the point C. We now move the section BC of the land curve up until B falls on A and C becomes the point D. The method then assumes that the composite curve is given by the sea curve as far as A, followed by the curve AD. The generalization of the method to more than one boundary is obvious.
Figure 6.1 is only diagrammatic, but it is chosen to represent roughly the case of $\lambda = 100\,\mathrm{m}$ considered above. The oversea curve is effectively the inverse-distance curve until the transition to the diffraction region takes place, whereas the over-land curve first goes over rapidly to the inverse-distance-squared curve before the transition to a diffraction curve much steeper than the oversea curve. It is not difficult to see that this method is not reciprocal, and in Figure 6.2 the curves for 20 km of sea followed by 20 km of land, and vice versa, are shown. The resulting fields at X and Y, respectively, differ by more than 20 db, the discrepancy being largely attributable to the extreme difference of the land and sea curves in the neighbourhood of the transmitter.
In one respect this empirical method fulfils our preconceived ideas in that at a sufficient distance from a boundary the field is characteristic of the section concerned as regards attenuation with distance. It is certainly wrong, however, in satisfying this condition immediately the boundary is passed, as we have seen that there must exist a disturbance function extending for some distance beyond the boundary.

On the assumption that the true field should lie somewhere between the values at X and Y in Figure 6.2, we are led by our previous arguments to consider their geometric mean (or arithmetic mean if the fields are expressed in decibels relative to some datum). In other words, we make the method strictly reciprocal by taking the geometric mean of the value it gives and the value it would give with the

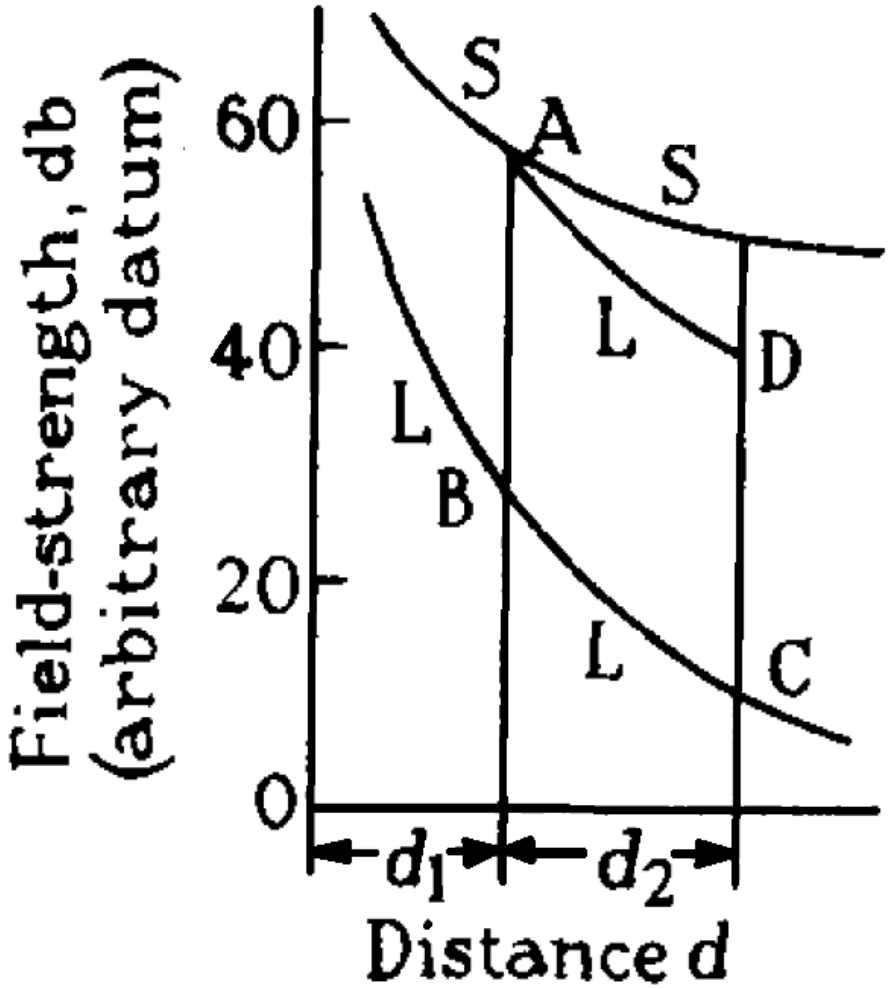

Figure 6.1.: Empirical method (P. P. Eckersley)

transmitter and receiver interchanged. In terms of our previous notation, the two fields in question for the general case of n sections would be

$$\mathfrak{E}_1(d_1) = \frac{\mathfrak{E}_2(d_1+d_2)}{\mathfrak{E}_2(d_1)} \cdot \frac{\mathfrak{E}_3(d_1+d_2+d_3)}{\mathfrak{E}_3(d_1+d_2)} \cdot \cdots \cdot \frac{\mathfrak{E}_n(d)}{\mathfrak{E}_n(d_1+\cdots+d_{n-1})} \tag{6.1}$$

and

$$\mathfrak{E}_n(d_n) = \frac{\mathfrak{E}_{n-1}(d_n+d_{n-1})}{\mathfrak{E}_{n-1}(d_n)} \cdot \cdots \cdot \frac{\mathfrak{E}_1(d)}{\mathfrak{E}_1(d_n+\cdots+d_2)} \,. \tag{6.2}$$

For the case when all the boundaries are in the diffraction regions of T and R, so that all the fields are of the type in Equation 3.1, these values reduce to

$$\frac{k_1}{\sqrt{d}} \cdot e^{\{-\alpha_1 d_1 - \alpha_2 d_2 - \cdots - \alpha_n d_n\}}$$

and

$$\frac{k_n}{\sqrt{d}} \cdot e^{\{-\alpha_1 d_1 - \alpha_2 d_2 - \cdots - \alpha_n d_n\}}$$

giving a geometric mean value of

$$\mathfrak{E}(d) = \sqrt{\frac{k_1 \cdot k_n}{\sqrt{d}}} \cdot e^{\{-\alpha_1 d_1 - \alpha_2 d_2 - \cdots - \alpha_n d_n\}} .$$

This is identical with the value in Equation 5.5 and hence with Equation 5.4. Thus, the method, when made reciprocal, gives the same result as we have already obtained under these conditions.

If we apply the method formally to a flat earth under conditions in which all the fields involved are of the inverse-distance-squared type given in Equation 3.4, the curves over any given section have the same shape, differing only in absolute magnitude depending on the value of ζ. Thus, the composite curve for transmission from T to R coincides with that for transmission over earth all of the type of the first section, giving a field-strength value of $\mathfrak{E}_1(d)$. The reciprocal geometric mean will then be $\sqrt{\mathfrak{E}_1(d) \cdot \mathfrak{E}_n(d)}$, giving under these special conditions the value in Equation 4.6 without the short-wave limit restriction. In general, however, the transmission paths are not confined to the Sommerfeld region, but may extend into the diffraction region, and alternatively in the Sommerfeld region itself the paths may not extend into the inverse-distance-squared region.

It is interesting to note that when there is a single boundary midway between T and R, we have to take the geometric mean

$$\mathfrak{E}_1(d_1) \cdot \frac{\mathfrak{E}_2(d)}{\mathfrak{E}_2(d_1)}$$

and

$$\mathfrak{E}_2(d_2) \cdot \frac{\mathfrak{E}_1(d)}{\mathfrak{E}_1(d_2)}$$

where $d_1 = d_2$, which is $\sqrt{\mathfrak{E}_1(d) \cdot \mathfrak{E}_2(d)}$. Thus, this particular result, which was shown to hold for the diffraction region, is given by the present method regardless of the shape of the $\mathfrak{E}_1(d)$ and $\mathfrak{E}_2(d)$ curves.

The method, as used by P. P. Eckersley, applied only to the Sommerfeld region and specifically to the case where $2\sigma \cdot \lambda c \gg \kappa$, but as a technique it is clearly general, as it can be regarded as a graphical process applicable whatever the shape of the sections of the curves from which the composite curve is constructed. It can therefore be used formally at all distances even when the transmitter or receiver may be close to a boundary.

Although there may be no theoretical justification for using the method in this way at the boundary, we have done so to obtain the transition region of the ground curve in Figure 5.4. We obtain the remarkable suggestion that on crossing the

boundary from land to sea there can be a considerable recovery of field-strength before the curve settles down to the over-sea type. We can see, however, that this result is not physically impossible in view of the large redistribution of energy that must take place below the fictitious sphere in the transition region to account for the change in type of the height-gain function. It appears that energy is fed from higher levels down to the ground, causing the signal strength to increase with distance for some way beyond the boundary.
If the boundary is at a distance d_1 from the transmitter and the receiver is at a distance d_2 beyond, the method gives for the field-strength $\mathfrak{E}(d_1 + d_2)$ the value

$$\mathfrak{E}(d_1 + d_2) = \sqrt{\mathfrak{E}_1(d_1) \cdot \frac{\mathfrak{E}_2(d_1 + d_2)}{\mathfrak{E}_2(d_1)} \cdot \mathfrak{E}_2(d_2) \cdot \frac{\mathfrak{E}_1(d_1 + d_2)}{\mathfrak{E}_1(d_2)}}\,.$$

In the diffraction region we have

$$\mathfrak{E}_1(d_1 + d_2) = \sqrt{\frac{d_1}{(d_1 + d_2)}} \cdot e^{-\alpha_1 \cdot d_2} \cdot \mathfrak{E}_1(d_1)$$

and

$$\mathfrak{E}_2(d_1 + d_2) = \sqrt{\frac{d_1}{(d_1 + d_2)}} \cdot e^{-\alpha_2 \cdot d_2} \cdot \mathfrak{E}_2(d_1)$$

so that

$$\frac{\mathfrak{E}(d_1 + d_2)}{\mathfrak{E}_1(d_1)} = \sqrt{\frac{d_1}{(d_1 + d_2)} \cdot \frac{\mathfrak{E}_2(d_2)}{\mathfrak{E}_1(d_2)}} \cdot e^{\left\{-\frac{1}{2} \cdot (\alpha_1 + \alpha_2) \cdot d_2\right\}} \tag{6.3}$$

giving the ratio of the field beyond the boundary to the field at the boundary.
As d_2 increases from zero under the conditions of Figure 5.4, $\mathfrak{E}_2(d_2)$ becomes much greater than $\mathfrak{E}_1(d_2)$, as these functions correspond to the field-strengths at a distance d_2 that is small and in the Sommerfeld region. Initially this effect more than compensates for the decrease due to the term $e^{\{-1/2 \cdot (\alpha_1 + \alpha_2) \cdot d_2\}}$, thus accounting for the rise in field-strength.
The field given by Equation 6.3 does not settle down to the characteristic type $e^{-\alpha_2 \cdot d_2}$ until d_2 exceeds the value $200 \cdot \lambda^{\frac{1}{3}}$, and $\mathfrak{E}_2(d_2)$ and $\mathfrak{E}_1(d_2)$ are given by their diffraction forms proportional to $e^{-\alpha_2 \cdot d_2}$ and $e^{-\alpha_1 \cdot d_2}$, when

$$\sqrt{\frac{\mathfrak{E}_2(d_2)}{\mathfrak{E}_1(d_2)}} \cdot e^{\left\{-\frac{1}{2} \cdot (\alpha_1 + \alpha_2) \cdot d_2\right\}}$$

becomes proportional to

$$e^{\{-\alpha_2 \cdot d_2\}}\,.$$

It was for this reason that the transition distance d_1 was taken to be $200 \cdot \lambda^{\frac{1}{3}}$ in constructing Figure 5.3. At the short-wave limit in the Sommerfeld region, d_1 is

given by $\frac{20}{\pi} \cdot \lambda \cdot |\zeta|^2$, i.e., the distance at which the inverse-distance-squared law comes into operation for the larger of the ζ values on the two sides of the boundary. If we use the method similarly in the neighbourhood of a boundary from sea to land, we obtain, instead of an increase, a correspondingly rapid drop in field-strength beyond the boundary before the curve settles down to a slope characteristic of overland transmission. Bearing in mind the fact that the reciprocity condition must be fulfilled, even in the neighbourhood of the boundary, we could explain away the proposed increase on the one hand only by a still more rapid drop on the other as the boundary is crossed. In order to obtain an idea of what the reciprocity condition implies, it is instructive to apply our method formally to construct a set of land-sea and sea-land curves for various positions of the boundary; the result for our example with $\lambda = 100\,\mathrm{m}$ is shown in Figure 6.3. It will be seen that, wherever the curves intersect, the reciprocity condition is obeyed. The land-to-sea recovery develops in the SOMMERFELD region as soon as the overland curve departs appreciably from the inverse-distance law and goes over to the inverse-distance-squared type. The effect becomes a maximum in the diffraction region where the limiting form given by Equation 6.3 is reached. On inspecting these curves, it is difficult to avoid the conclusion that there must be some such recovery if the reciprocity condition is to be met everywhere, though admittedly this does not justify the method we have adopted to obtain the curves. Rather, we must regard the method as a tentative approach to the study of the disturbance function in an attempt to reveal the general features we should expect the rigid solution to present.

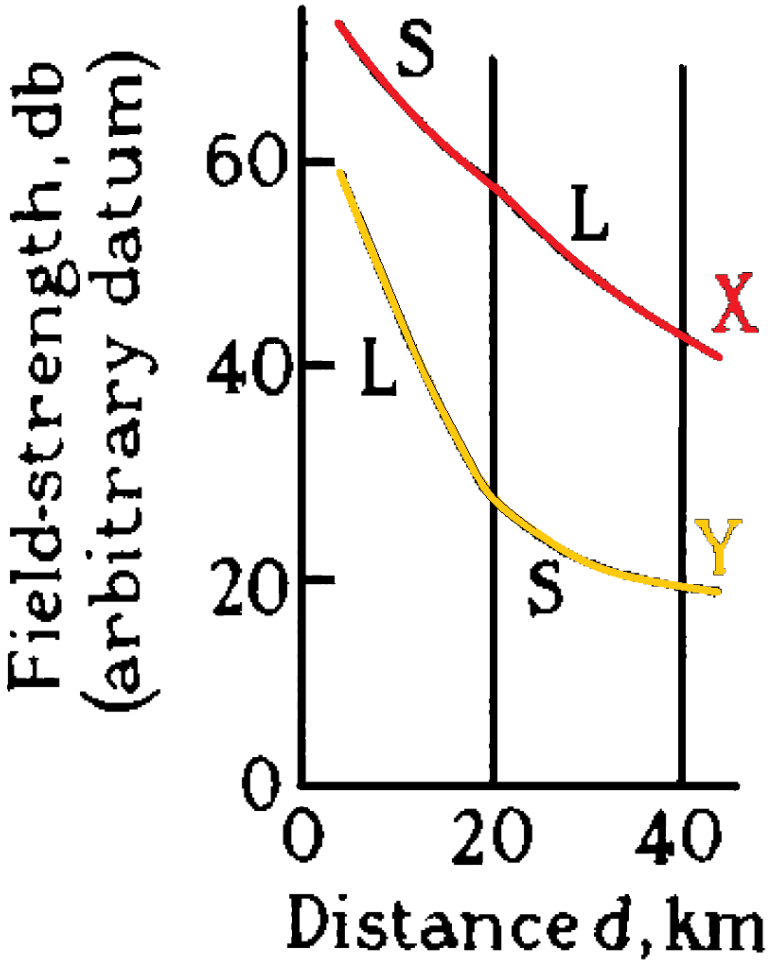

Figure 6.2.: Non-reciprocity of the P. P. Eckersley method. X and Y represent fields

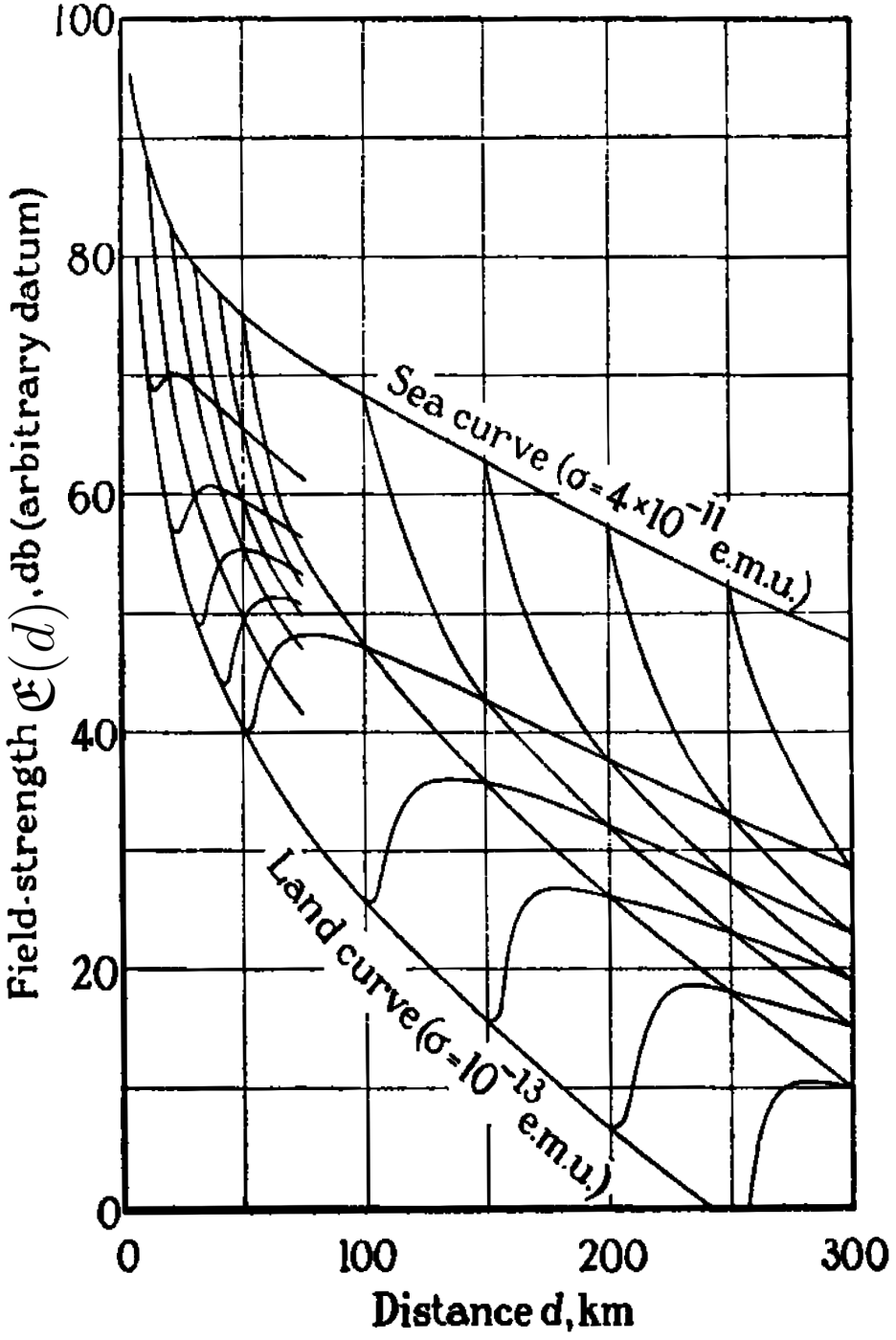

Figure 6.3.: Field-strength/distance curves for various positions of a land-sea or sea-land boundary, obtained by the reciprocal empirical method. $\lambda = 100\,\mathrm{m}$

7. Experimental Evidence

It is natural to ask to what extent our conclusions are borne out by experiment, especially with regard to the land-to-sea recovery. Unfortunately, it appears that practically no specially controlled experiments have been carried out across a land-to-sea boundary, so that the issue may still be regarded as open. The author is indebted, however, to the B.B.C. for permission to refer to the fact that they have found indications of a recovery in field-strength where a wave has crossed from ground of low to ground of medium conductivity. Figure 7.1 is a reproduction of one of their curves obtained by measurements on a test transmitter at Start Point ($\lambda = 286\,\mathrm{m}$) taken along a line to Happisburgh on the Norfolk coast. It is known that there are local regions of poor conductivity in East Anglia, and it will be noticed that the recovery is preceded by an especially steep section of the curve. The fact that the recovery is small may be accounted for partly by the relatively small change in conductivity involved, and partly because the change may not be sudden, as in our idealized picture, or extend far enough for the effect to be fully developed. By itself, therefore, it can throw little light on the order of accuracy of the empirical method, though it supports the contention that a recovery of some kind is to be expected. It draws attention to the need for a specific experiment under controlled conditions across a land-sea boundary where the sea stretches well into the diffraction region with respect to the boundary.[1]

The B.B.C. experiments also support the view upon which our argument has been based, that the modification to the field-strength is negligible until the boundary is reached. In the Start Point experiments, for instance, they have found that the initial transmission across the sea gives a field at the Dorset coast agreeing with the calculated all-oversea value. The curve in Figure 7.1 also shows an increased rate of drop in field-strength crossing a sea-to-land boundary, the crosses shown being calculated according to our method, assuming $\sigma = 1 \times 10^{-13}$ e.m.u. for the land, though this evidence is not conclusive without a local survey to justify the use of this average value for the conductivity. Further B.B.C. experiments with a transmitter at Rampisham near Dorchester do, however, support this choice of σ for the initial part of the land path in the Start Point tests.

Other B.B.C. measurements agree with those of T. L. Eckersley, quoted above, in supporting the proposed solution over part-land and part-sea routes where the detailed nature of the field in the neighbourhood of a boundary is not under

[1] A predicted recovery of 12 dB has since been obtained on a wavelength of $\lambda = 4\,\mathrm{m}$ – G. M. (10.12.48).

consideration. The surveys which they have carried out in connection with the service areas of their various medium-wave stations also throw valuable light on the problem of differing conditions along different radials from a transmitter. In Figure 7.2 are shown some field-strength contours round the transmitter at Washford Cross, and it will be seen how profound an effect the estuary of the River Severn has on the shape of the contours.

These contours suggest that our idealizing assumption for working out the field-strength variations along a radial is roughly true, and though the contours of equal field-strength must be rounded off where they cross a land-sea boundary, they do in fact undergo very rapid changes in direction. Even where the wave is transmitted almost parallel to a coast-line, the method gives a good indication of the variation of field-strength inland from the coast at different distances from the transmitter, as, for instance, has been found in the case of the service area of Start Point along the South Coast towards Brighton.

This aspect of the problem is obviously associated with the size of the distances concerned relative to the wavelength, and it is here that the treatment of GRÜNBERG and FEINBERG may be of special importance. There also arises the allied question of coastal refraction, but this is a subject we cannot enter into here, where we have been considering only the field-strength, though it is of equal concern to those interested in the application of medium and long waves to navigational aids.

There is one further point on which it would be most useful to have experimental evidence. According to the curves in Figure 6.3, it should be possible under some conditions for a transmitter radiating initially over sea to produce a stronger signal over the succeeding land than it would if it were placed on the land itself.

For instance, we see that a transmission over 50 km of sea and then over 50 km of land gives a field-strength about 7 dB greater than for a transmission over 50 km of land alone, the reason presumably being that the initial sea path has reduced the heavy earth losses in the neighbourhood of the transmitter.

An experiment to investigate this possibility would help greatly towards proving whether our method of representing the field in the neighbourhood of a boundary is reasonably accurate or not. In any case, the reciprocity requirement emphasizes a point that is not always fully appreciated, that not only should the wave be given a good start by placing the transmitter on a site of high conductivity, if it is to be used to the best advantage, but also the receiving site should be made equally conducting. In other words, the earth losses are important not only in the neighbourhood of the transmitter but also at the receiver, a fact which is implied in the identical form of the height-gain factors $f(h_T)$ and $f(h_R)$.

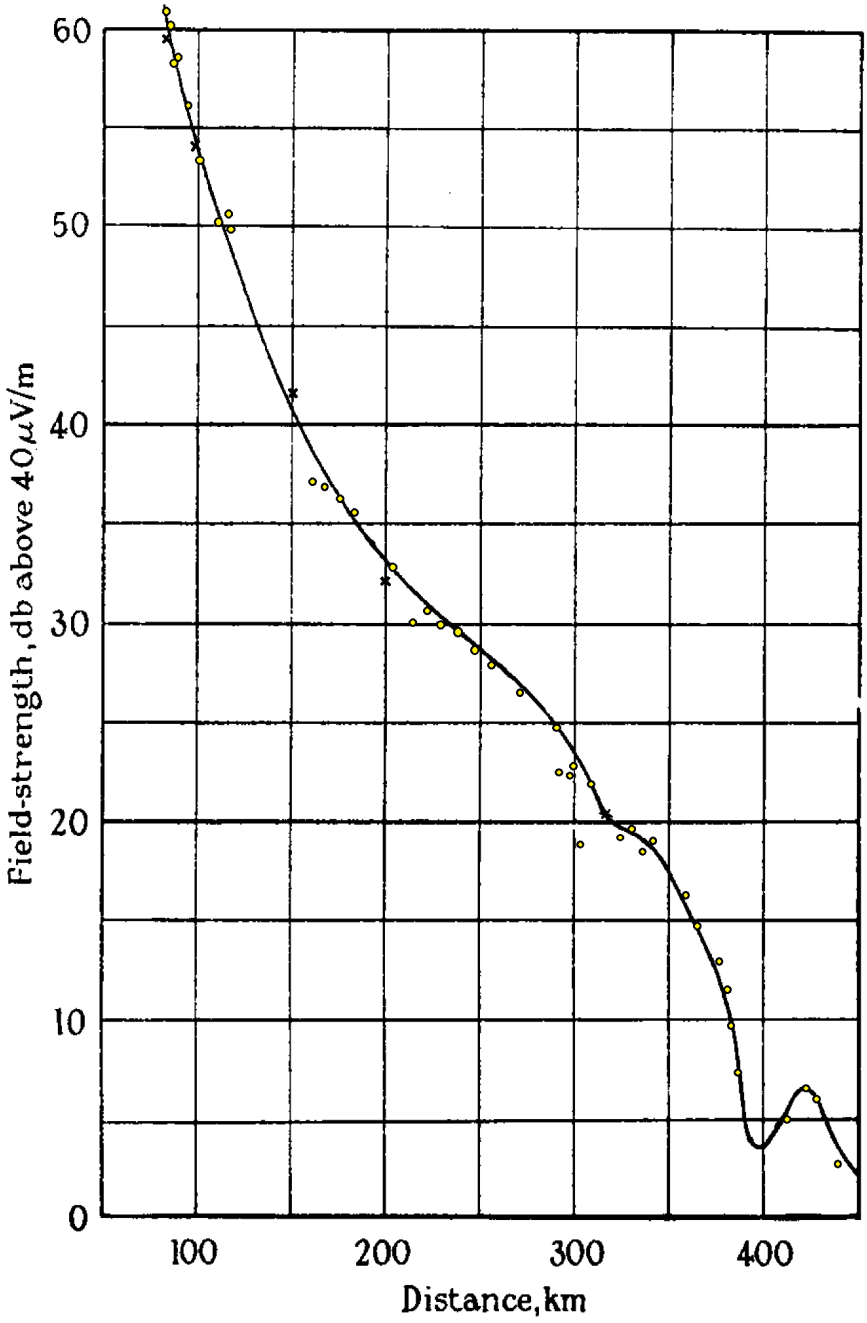

Figure 7.1.: Experimental curve over inhomogeneous land, showing a small recovery effect.

$\lambda = 286\,\mathrm{m}$

ooo correspond to observed values

××× correspond to calculated values

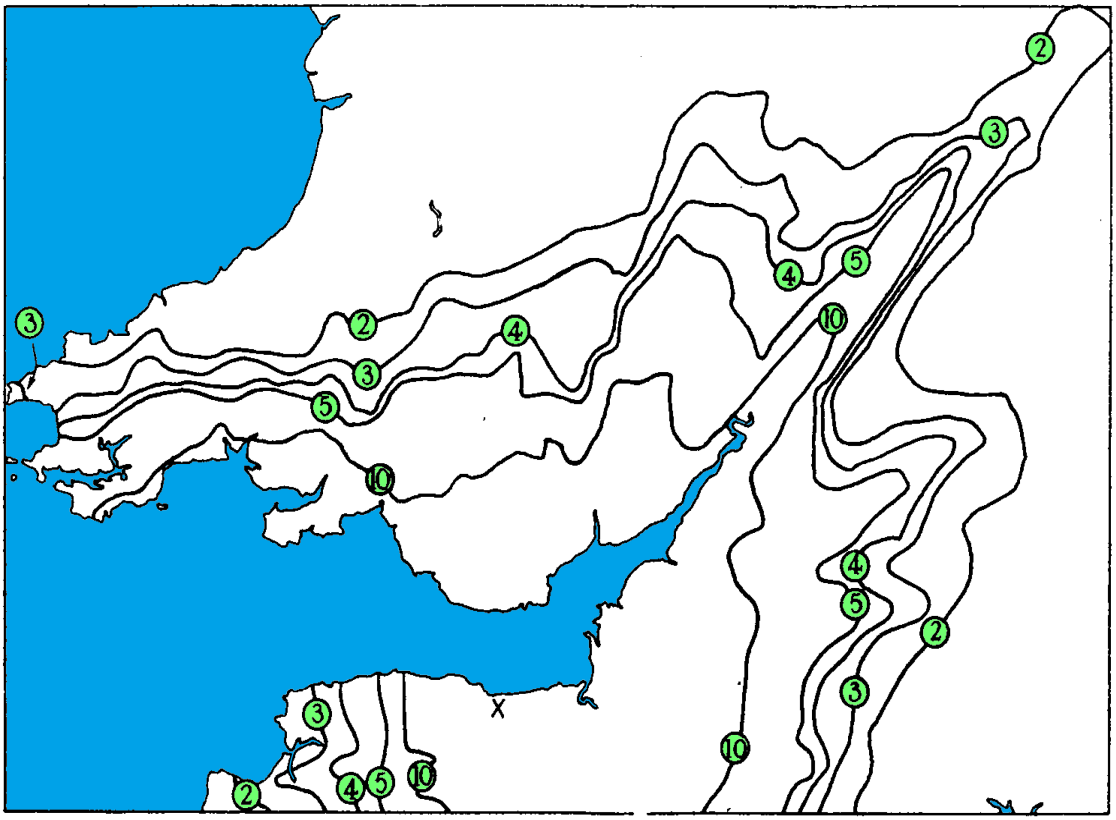

Figure 7.2.: Experimental field-strength contours (in $\mathrm{mV\,m^{-1}}$), showing the radial effect.

$\lambda = 371.1\,\mathrm{m}$
Power $= 70\,\mathrm{kW}$

8. Practical Considerations

There remains the problem of interpreting the method we have discussed in a practical form for the computation of the field-strength for any given route of two or more sections. We have been primarily concerned with the development of the fundamental ideas upon which the method is based in an attempt to reveal the features that a complete solution would present, so that we cannot enter here into great practical detail. We will, however, conclude with a few general suggestions that may be useful to the engineer.

Even if we do not wish to employ the method in the neighbourhood of a boundary where it is admittedly speculative, the fact that it agrees with our earlier arguments for the regions well beyond the boundary enables us to base our method of computation on Equation 6.1 and Equation 6.2 rather than on Equation 5.4. This is in general preferable because all the quantities involved are directly obtainable from standard ground-wave propagation curves for various conductivities, and it is not limited strictly to the diffraction region. There is not space here to present such ground-wave curves, and it must be assumed that reference can be made to existing sets. It is desirable to use curves provided with a decibel scale relative, for example, to $1\,\mu\mathrm{V\,m^{-1}}$ as datum, so that the computation of Equation 6.1 and Equation 6.2 and their geometric mean can be turned into a purely additive process.

If a number of individual and unrelated cases have to be worked out, it is probably simplest to compute each one in turn directly with the aid of Equation 6.1 and Equation 6.2 expressed in decibel form. Suppose, however, that a detailed investigation has to be made of the field-strength contours round a given transmitter on a fixed wavelength, so that the field-strength/distance curves have to be constructed along a large number of radials. It is then probably worthwhile to adopt the following procedure.

We make use of the fact that the standard field-strength curves on a linear scale of distance eventually become effectively linear to extrapolate the linear portion back to the ordinate $d = 0$, where it will make an intercept $2p$, as in Figure 8.1, on the decibel scale. If the slope of this straight line is q decibels per unit distance, the field-strength at a distance d in decibels can then be written in the form

$$\mathfrak{E}(d) = 2p - qd + 2 \cdot u(d)\,,$$

where $2 \cdot u(d)$ is the departure of the actual curve from the straight line in Figure 8.1. This function $u(d)$ tends to zero with increasing d, and to infinity as d approaches

0. If, however, we have two types of earth for which separately we have

$$\mathfrak{E}_1(d) = 2p_1 - q_1 d + 2 \cdot u_1(d)\,, \tag{8.1}$$

and

$$\mathfrak{E}_2(d) = 2p_2 - q_2 d + 2 \cdot u_2(d)\,, \tag{8.2}$$

then $\mathfrak{E}_1(d)$ and $\mathfrak{E}_2(d)$ both tend to the inverse-distance value as d approaches 0. It therefore follows that

$$u_2(d) - u_1(d) \to p_1 - p_2$$

as $d \to 0$. Thus, if we define a function ${}_2u_1(d)$ by

$${}_2u_1(d) = u_2(d) - u_1(d)\,, \tag{8.3}$$

it will approach a finite limit $(p_1 - p_2)$ as d approaches 0, and will tend to zero as d increases.
In terms of this notation, our required $\mathfrak{E}(d)$ over a path of n sections may be reduced to the form

$$\begin{aligned}\mathfrak{E}(d) = {} & p_1 + p_n + {}_1v_n(d) - [q_1d_1 + q_2d_2 + \cdots + q_nd_n] \\ & + [{}_1u_2(d_1) + {}_2u_3(d_1 + d_2) + \cdots + \qquad (8.4)\\ & + {}_{n-1}u_n \cdot (d_1 + d_2 + \cdots + d_{n-1}) \\ & + {}_nu_{n-1}(d_n) + {}_{n-1}u_{n-2} \cdot (d_n + d_{n-1}) \\ & + \cdots + {}_2u_1 \cdot (d_n + d_{n-1} + \cdots + d_2)]\,, \qquad (8.5)\end{aligned}$$

where

$${}_1v_n(d) = u_1(d) + u_n(d)\,. \tag{8.6}$$

From Equation 8.3 we have

$${}_1u_2(d) = -{}_2u_1(d)\,,\ \text{etc.}, \tag{8.7}$$

while from Equation 8.6 it follows that ${}_nv_1(d) = {}_1v_n(d)$, and from Equation 8.1 and Equation 8.2 that

$$p_1 + p_n + {}_1v_n(d) \to I(d) \tag{8.8}$$

as $d \to 0$. $I(d)$ is the inverse-distance value.
From the relations in Equation 8.7 and Equation 8.8 we can see that $\mathfrak{E}(d)$ in Equation 8.4 approaches, as it should, the inverse-distance value $I(d)$ when the total distance d is indefinitely small.
The advantage of this method of computation is that in the term $[q_1d_1 + q_2d_2 + \cdots + q_nd_n]$ any sections with the same q value can be grouped together as a single section of the same total length, while the u functions are in general small corrections which are zero when the boundaries concerned are at a sufficient distance from the

terminal with respect to which they are reckoned. The function ${}_1v_n(d)$ involves only the total distance, and is zero in many practical cases.

This method implies, as a preliminary, that we should prepare from the standard propagation curves for the types of earth involved a table of constants of the form $(p_1 + p_n)$, a set of straight lines through the origin giving $q_1 d$, $q_2 d$, etc., for the computation of $[q_1 d_1 + q_2 d_2 + \cdots + q_n d_n]$, and curves of the various functions ${}_2u_1(d)$, etc., and ${}_1v_n(d)$. Very often, in practice, there are only two types of earth involved, e.g., land and sea, though sometimes the land may have to be subdivided, e.g., into medium conductivity ($\sigma = 1 \times 10^{-13}$ e.m.u.) and poor conductivity ($\sigma = 1 \times 10^{-14}$ e.m.u.), so that the method is not so complicated as it may seem at first sight.

As an example of the basic curves derived from the standard propagation curves, those for $\lambda = 100$ m are shown in Figure 8.2 for a part land and part sea route where for land σ is everywhere 1×10^{-13} e.m.u. In this case the constants required are $2p_L$ and $2p_S$ for routes both beginning and ending with land and with sea respectively, and $p_L + p_S$ where there is land at one end of the route and sea at the other. Their values will depend upon the radiated power and the datum level for the field-strength decibel scale, whereas the functions shown in Figure 8.2 are independent of them, being essentially decibel differences by definition.

We cannot enter further here into the details of this method of computing the field-strength, but it is found to lead to an easy and rapid technique after a little experience in handling of it has been acquired.

As an example of the method, a field-strength/distance curve is given in Figure 8.3 for a 1 kW transmission on a wavelength of 1000 m over 200 km of land ($\sigma = 1 \times 10^{-13}$ e.m.u.), 200 km of sea, 200 km of land ($\sigma = 1 \times 10^{-14}$ e.m.u.) followed by sea to an indefinite distance. On this wavelength the land-sea recovery in field-strength is not so pronounced, owing to the approach towards the long-wave limit, but beyond the boundary at 600 km the field-strength holds up for 300 km, and the curve illustrates well the practical implications of the solution here proposed. It is clearly a matter of great importance to the engineer that the problem should be subjected to further theoretical and experimental study.

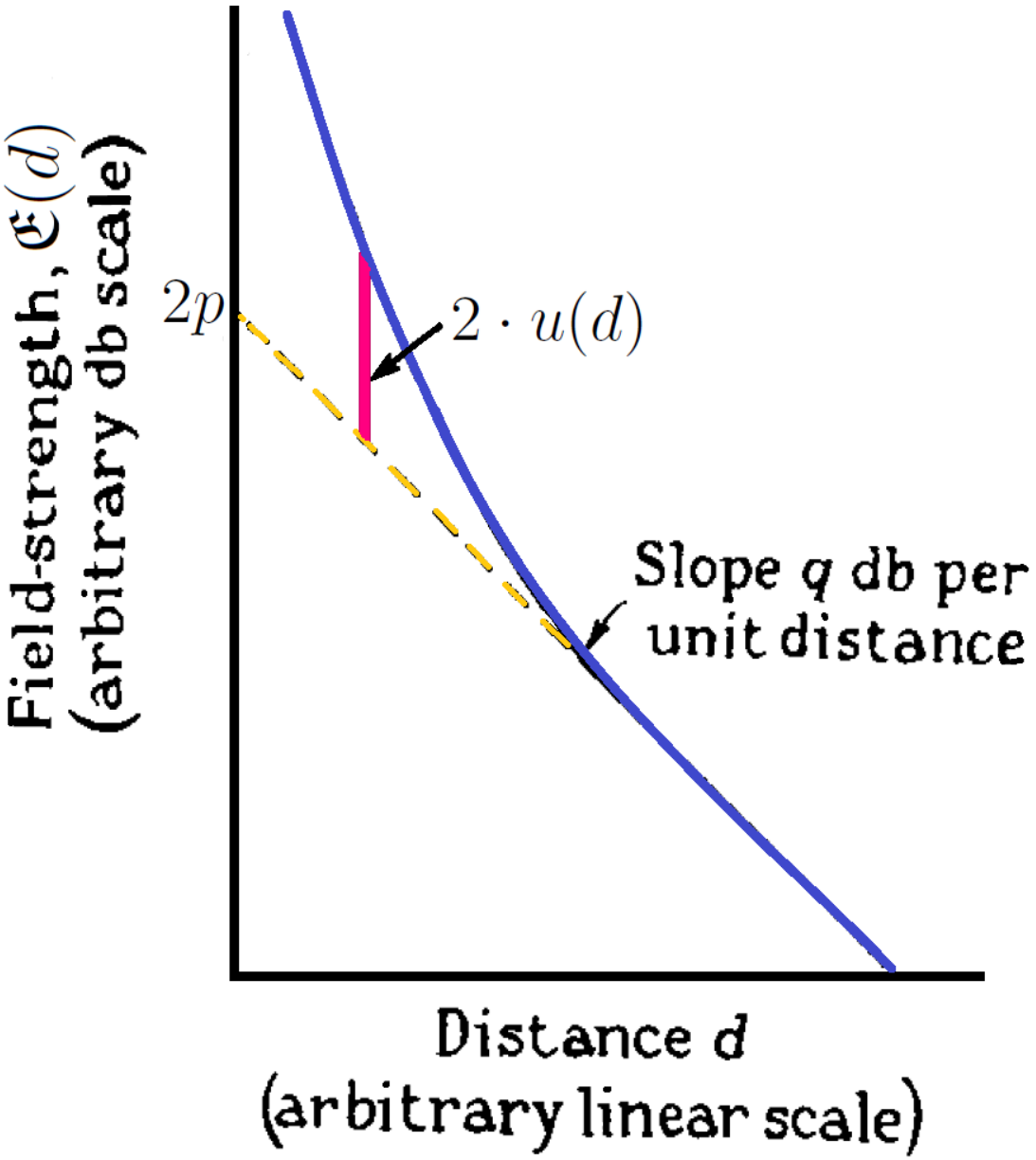

Figure 8.1.: Practical resolution of a field-strength/distance curve.

$$\mathfrak{E}(d) = 2p - qd + 2 \cdot u(d)$$

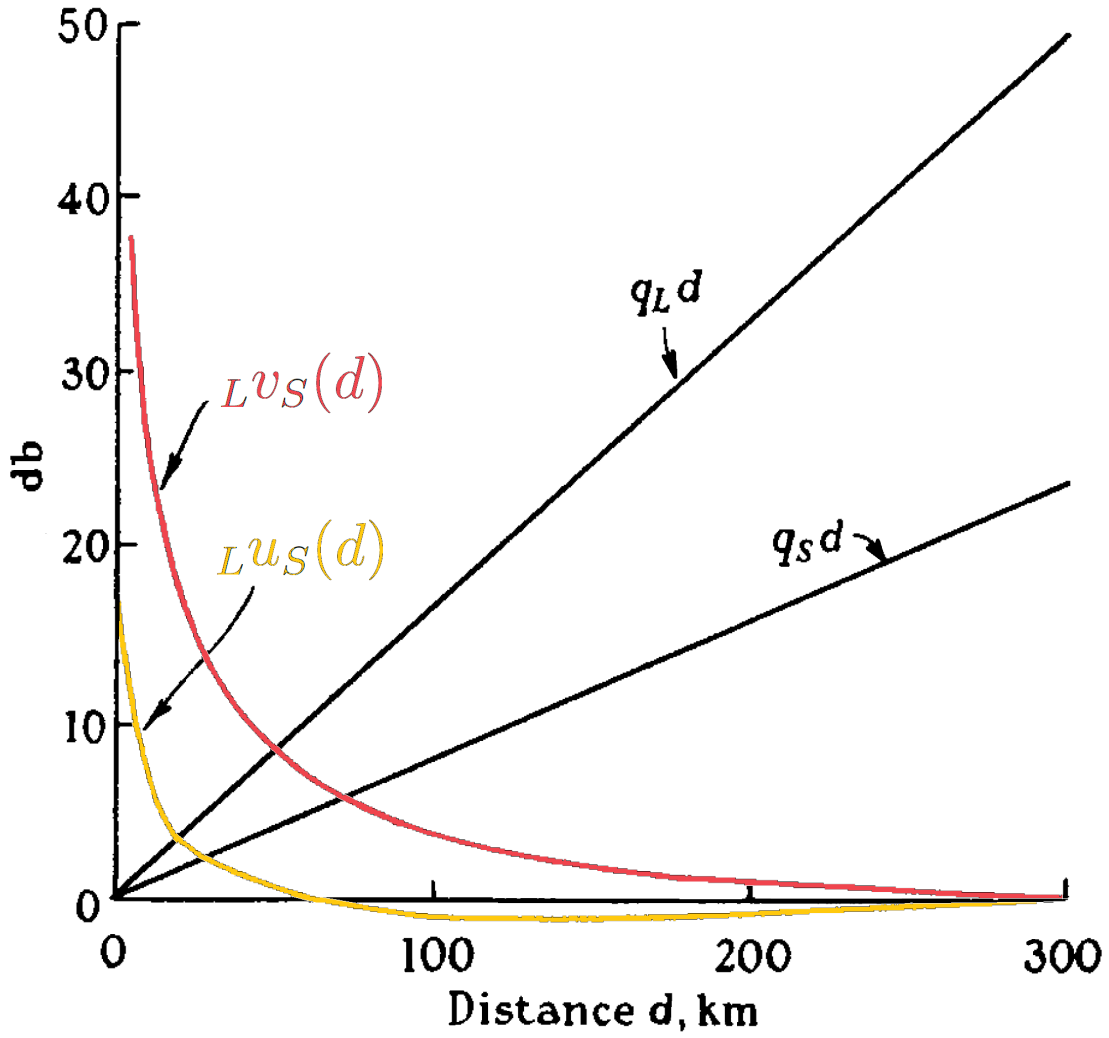

Figure 8.2.: Basic curves for use with routes having various combinations of land and sea sections for a given wavelength.

$\lambda = 100\,\mathrm{m}$
$\sigma = 1 \times 10^{-13}$ e.m.u. (land) and $\sigma = 4 \times 10^{-11}$ e.m.u. (sea)
$_L u_S(d) = u_L(d) - u_S(d)$
$_L v_S(d) = u_L(d) + u_S(d)$

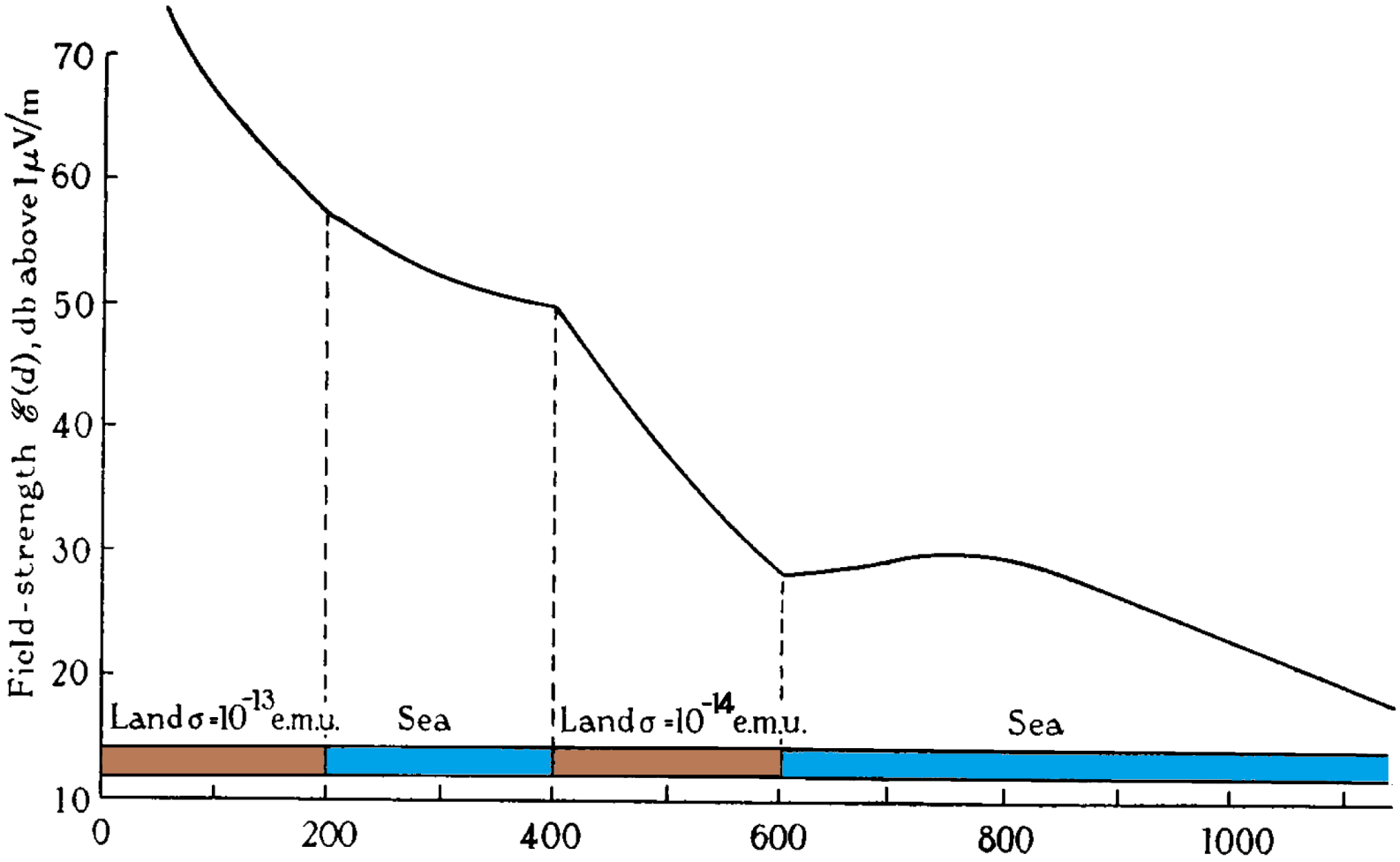

Figure 8.3.: Practical example showing several boundaries.

$\lambda = 1000\,\mathrm{m}$
Power $= 1\,\mathrm{kW}$

9. Conclusions

In conclusion it is necessary to stress again the imperfections of the solution. It is difficult to assess the approximations involved in the argument based on the properties of the wave above the fictitious sphere, though it is felt that the physical picture it presents is essentially sound, even if it needs modifying in detail. We can therefore feel justified in using it to support an empirical method that gives the same result under the same conditions. The extension of this method to the neighbourhood of a boundary is admittedly speculative, but it has served to draw attention to some of the features we may expect a rigid solution to reveal.

The inadequacy of the method is most clearly shown in the fact that no consideration is taken of the phase of the wave at the boundary. It therefore leaves unanswered the important question of coastal refraction and the distortion of the surfaces of constant phase, for which a complete solution of the boundary problem is needed. It is interesting that the formula for ground-to-ground propagation has been based on a study of the height-gain effects, but this is reasonable in view of the redistribution of energy that must take place above a boundary.

In the diffraction region, the appropriate height-gains can be applied for a raised transmitter or receiver in accordance with the picture that has been given, but the empirical method that has been applied formally only to the ground-to-ground case can be extended to study the effect of raising a terminal that is near a boundary. In many applications of the theory to medium- and long-wave propagation the height-gain effects are negligible, though it should not be overlooked that quite considerable effects may be involved in the case of high-flying aircraft employing medium-wave navigational aids.

It is hoped that this paper may stimulate others to pursue this most important subject on the theoretical side, and to carry out controlled experiments designed to test the views that have been put forward.

10. Acknowledgments

The author is indebted to MARCONI's Wireless Telegraph Co., Ltd., for permission to publish the paper and to members of the Research Division for help and encouragement in its preparation. He also gratefully acknowledges the co-operation received from the research staff of the British Broadcasting Corporation and the permission to publish the experimental results given in Figure 7.1 and Figure 7.2 and described in chapter 7.

Part IV.

Fourth Paper

Ground-Wave Propagation over an Inhomogeneous Smooth Earth. Part 2. Experimental Evidence and Practical Implications

(The paper was first received 20th June, and in revised form 1st September, 1949.)

Abstract

This paper is a sequel to an earlier one and recapitulates its main points, in particular the recovery in field-strength that can occur on crossing land/sea boundary. Further experimental evidence is reviewed which is in agreement with the theory but is not sufficiently strong to be conclusive. The optimum conditions for obtaining the recovery in practice are discussed, favouring the choice of a wavelength of about 4 m where the distances required are quite small, and of 100 m where the attenuation with distance in the diffraction region is much greater over land than over sea. Experiments on approximately these two wavelengths are then described which give the expected increases in field strength and show that the calculated curves fit the observations as closely as the experimental conditions warrant. In the light of this confirmation of the theory, some of its practical implications are reviewed, in particular with regard to the siting of transmitters near stretches of water in relation to their service areas, the possibilities of frequency-sharing under ground-wave conditions, and the ground-wave coverage of medium- to long-wave navigational aids.

1. Introduction

The earlier paper by one of the authors will be referred to here as Part III. Its main purpose was to provide a reasonable theoretical basis for a proposed method of calculating groundwave field-strengths over composite paths, e.g. part land and part sea, different or where there are well-defined sections of land of conductivities. One of the guiding principles in formulating the solution was the satisfaction of the reciprocity condition which had been neglected in an empirical solution proposed many years ago by P. P. Eckersley, [8]. It was shown that this method, when made reciprocal by taking a geometric mean, gives the same result as that derived by a theoretical argument based on height-gain considerations for points well away from the boundaries.

In order to obtain an idea of the implications of reciprocity near a boundary, it was formally assumed that the P. P. Eckersley method made reciprocal is valid even right up to a boundary. When this was done, it yielded the striking suggestion that when a vertically polarized ground-wave crosses from land to sea there can be a considerable recovery in field-strength, with a correspondingly rapid drop when the boundary is crossed in the reverse direction. Theoretically, for horizontal polarization these effects should be reversed, but in practice they would be confined close to the earth where the field would in any case be very weak.

The amount and extent of the recovery effect can be remarkable and, if real, of great practical importance. In view of the admittedly empirical nature of the solution in this vital region, it is obviously highly desirable to test it experimentally. It was pointed out in Part III that the existing evidence from published field-strength surveys is scanty in this respect owing to the complexity of the transmission paths concerned. The most that can be said is that it does not disprove the suggested method, since in none of the cases are conditions such as are required to reveal the effect unequivocally.

In the next Section reference is made to some work that has come to the notice of the authors since Part III was published. While it contains further evidence of the recovery, the results are not conclusive enough to convince the engineer of the striking effects that are predicted by the theory under the optimum conditions. It is clearly requisite to make some experiments under carefully controlled conditions for which the predicted recovery is greatest, and it is one of the purposes of the present paper to describe two such experiments carried out since Part III was written.

The techniques used are of sufficient intrinsic interest to merit a fairly detailed description, and in the light of the confirmation afforded of the proposed method

some practical implications of its use in ground-wave propagation are discussed. These include the siting of transmitters relative to stretches of water, the problem of frequency-sharing for ground-wave working in the short and medium wavebands, and ground-wave coverage areas free from ionospheric interference on long waves.

2. Further Existing Evidence

A paper has been recently published by KIRKE, [22], in which he analyses the results obtained in a number of tests made by the B.B.C., and uses them to compare the above method, which he calls the MILLINGTON method, with the P. P. ECKERSLEY method and one due to T. SOMERVILLE of the B.B.C. As KIRKE admits, the latter has no theoretical basis, but from general principles it is a step in the right direction, so that it often satisfies the reciprocity condition approximately, though it does not do so of necessity. The B.B.C. tests were carried out over land following an initial sea path, and the SOMERVILLE method can give a very rapid drop immediately beyond the boundary. Under no conditions does it give a recovery effect, though it could do so if it were made formally reciprocal in the way that the MILLINGTON method makes the P. P. ECKERSLEY method reciprocal. There is, however, no corresponding theoretical justification for so doing.
Perhaps the most interesting results discussed by KIRKE are those obtained by the Danish Department of Posts and Telegraphs, [1]. Here there are definite indications of recovery effects, but it is unfortunate that no measurements were made along the sea paths. In a private communication from Mr. G. BRAMSLEV, the authors have been informed of a further test on 1176 kHz ($\lambda = 255$ m) in which measurements were made for 14 km out to sea beyond a boundary 76 km from the transmitter, with a single reading at a further coast 27 km across the sea. The results are shown in Figure 2.1 with the curve obtained by the MILLINGTON method assuming a land conductivity of 1×10^{-13} e.m.u. in accordance with the surveyed value. The agreement is good, but on this wavelength the contrast between the land and sea conductivities would have to be greater to show a really well-defined recovery effect so near the transmitter. The hold-up in field-strength is rather similar to that predicted on a wavelength of 1000 m with a land conductivity of 1×10^{-14} e.m.u. shown in Figure 8.3.

The authors are indebted to Dr. VAN DER POL for drawing their attention to some experiments done many years ago, in which a field-strength survey was carried out round a transmitter on a wavelength of $\lambda = 299$ m at Hilversum, [31]. Figure 2.2 shows the contours in the neighbourhood of Amsterdam, and Figure 2.3 gives the field-strength/distance curve along the radial passing through the centre of the town. It will be seen that the heavy attenuation in crossing the town is followed by a small recovery effect. It would be interesting to know in this case, where there is a comparatively localized area of high absorption, to what extent the filling-in beyond is due to energy that has flowed round the sides of the obstruction

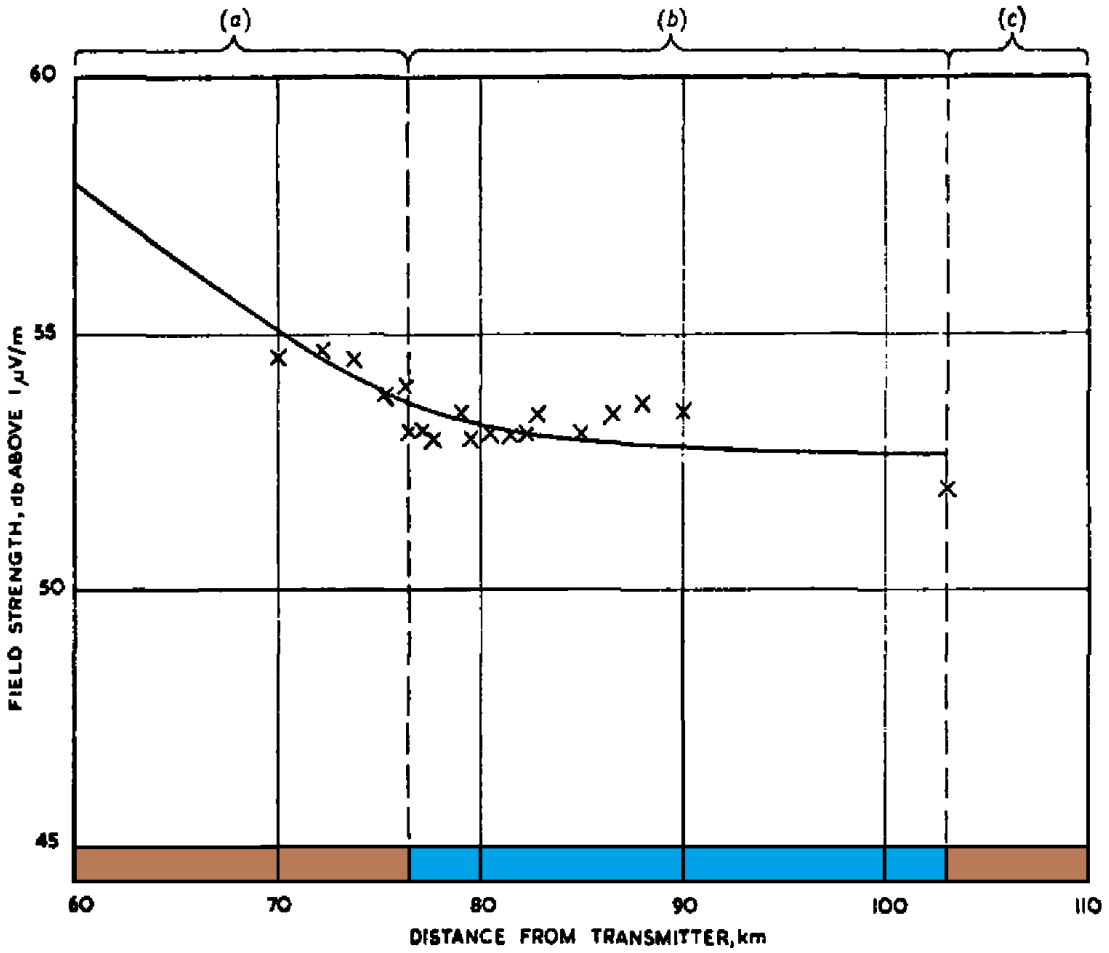

Figure 2.1.: Comparison of the Millington method with the results of Bramslev on 1176 kHz.

× × Experimental values
Frequency 1176 kHz

(a) Land, $\kappa = 5$ e.s.u., $\sigma = 1 \times 10^{-13}$ e.m.u.
(b) Sea, $\kappa = 80$ e.s.u., $\sigma = 4 \times 10^{-11}$ e.m.u.
(c) Land, $\kappa = 5$ e.s.u., $\sigma = 1 \times 10^{-13}$ e.m.u.

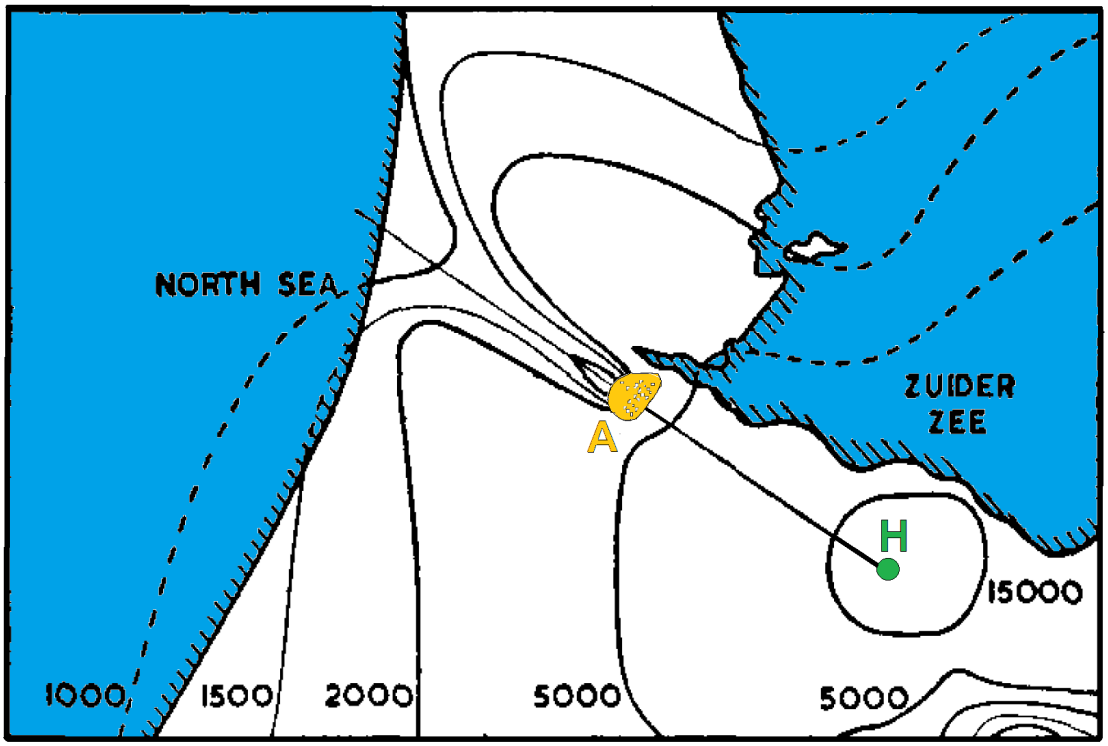

Figure 2.2.: Field-strength contours near Amsterdam from Hilversum (After VAN DER POL).
Wavelength $\lambda = 299\,\mathrm{m}$
H = Hilversum
A = Amsterdam
Contours in $\mu\mathrm{V\,m^{-1}}$

rather than from above, or whether it can be explained in terms of the theory of Part III, treating each radial on its own merits.

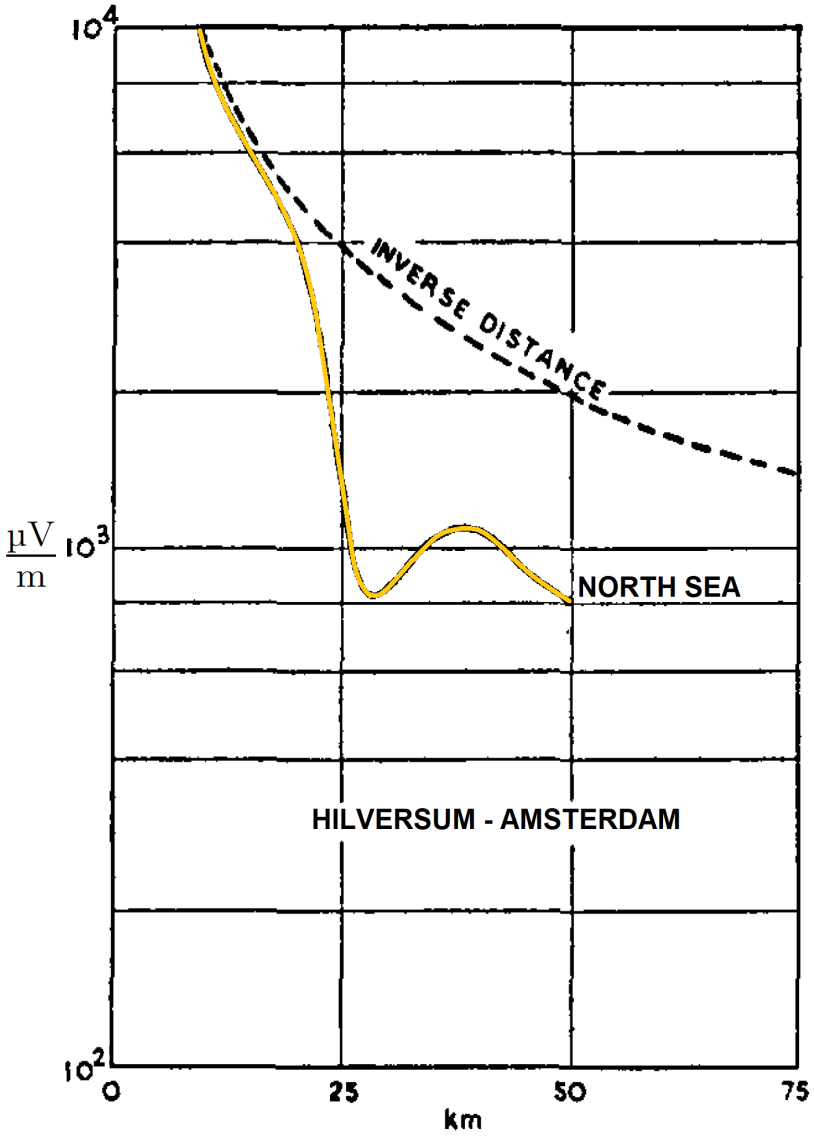

Figure 2.3.: Field-strength/distance curve, Hilversum-Amsterdam (After VAN DER POL) showing recovery effect.
Wavelength $\lambda = 299\,\mathrm{m}$

3. The Optimum Conditions for Obtaining the Recovery Effect

Before proceeding to the description of the experiments that were specifically designed to investigate the recovery effect, it will be useful to outline the considerations which led to the choice of wavelengths and transmission paths. It was pointed out in Part III that at the long-wave limit both land and sea behave as if they were perfect conductors, and that the ground-wave propagation should be independent of any changes in conductivity along the path. At the short-wave limit, however, where the earth is dielectric, the different permittivities of land and sea imply different initial height-gain functions, even though the land and sea curves have the same slope in the diffraction region. The theory given in chapter 3 and chapter 4 shows that the energy redistribution at the short-wave limit takes place in a layer extending from the earth's surface to a height h_1, where $\frac{2\pi \cdot h_1}{\lambda \cdot \zeta} \gg 1$ and ζ, defined by Equation 3.2a and Equation 3.2b, respectively, is the greater of the two earth-constant functions for the sections meeting at the boundary. As a working value, h_1 may be defined by $\frac{1}{2} \cdot \lambda \cdot |\zeta|$. Moreover, the effect will be well developed, provided the boundary is at a distance of at least $\frac{20 \cdot \lambda \cdot |\zeta|^2}{\pi}$ from the transmitter, where ζ is here the smaller of the two earth constant functions. An obvious advantage of using a sufficiently short wave is that the distances involved can be quite small, making it practicable to use a low-power portable transmitter and to find a convenient and suitable site. On the other hand, with an increasingly short wavelength it becomes difficult to find a site that is free from irregularities of terrain that are comparable with the wavelength, and as the height h_1 decreases, the effect becomes too critical with height.

A preliminary consideration suggested that a wavelength of about 4 m would be an almost ideal compromise. At this wavelength the function ζ for sea is controlled mainly by the conductivity and has a numerical value of 30, so that the corresponding value of h_1 is 60 m. The corresponding distance $\frac{20 \cdot \lambda \cdot |\zeta|^2}{\pi}$ for land depends mainly upon the permittivity, but it is certainly less than 1 km. For an assumed permittivity $\kappa = 15$ e.s.u., the maximum possible recovery given by the MILLINGTON method would be about 15 dB in this case, and even within a kilometre or so of the transmitter it should be 12 dB and occur at only 0.5 km beyond the boundary.

These conditions are obviously suitable for a test of the theory, and in view of them an experiment was planned and carried out. The result confirming the prediction

was mentioned in a footnote added during the final proof-reading of Part III, and a brief preliminary notice was published elsewhere, [24]. A more detailed account is given in the next Section, including a description of the technique used for the checking of the reciprocity condition. The success of this experiment under the conditions prevailing at the short-wave limit is not a conclusive proof in favour of the theory at longer wavelengths where the attenuation with distance in the diffraction region varies with conductivity. The analysis for this case, given in chapter 5, while based on a physical picture of the energy flow above the earth, was not essentially reciprocal as in the case for the short-wave limit in chapter 4, but had to be formally made so, leaving an element of doubt with regard to the magnitude of any approximation involved.

It was therefore decided to carry out a further test on a wavelength where the contrast between the land and sea attenuation in the diffraction region is greatest. The choice of wavelength depends upon the conductivity of the land, and, bearing in mind that the conductivity of much of England is of the order of 1×10^{-13} e.m.u., theoretical considerations favour the choice of a wavelength of about 100 m. It will be seen from Figure 6.3, which was constructed for these conditions, that the maximum recovery to be expected is of the order of 10 dB and that it is already effectively attained when the boundary is 100 km from the transmitter. In order to show the curve settling down to the characteristic slope of the sea attenuation, a total sea path of at least 100 km would be needed.

These requirements are consistent with the likelihood of finding a suitable route and with them the obtaining of an adequate field-strength at the boundary from a reasonably high-powered transmitter. It is known, however, that even in the daytime the ionospheric reflection can be comparable with the ground-wave signal at a distance of 100 km overland on a wavelength of 100 m, so that it would be essential to transmit pulses in order to be able to isolate the ground wave.

Such an experiment has been carried out and, at the time of writing, preliminary notice of the results has been sent for publication, [25], in view of their immediate practical importance. It is described in chapter 5 with an account of the technique, which naturally differed greatly from that used for the short-wave experiment. Readers not interested in the technical details may wish to refer directly to the results shown in Figure 3.1, Figure 3.2 and Figure 5.3.

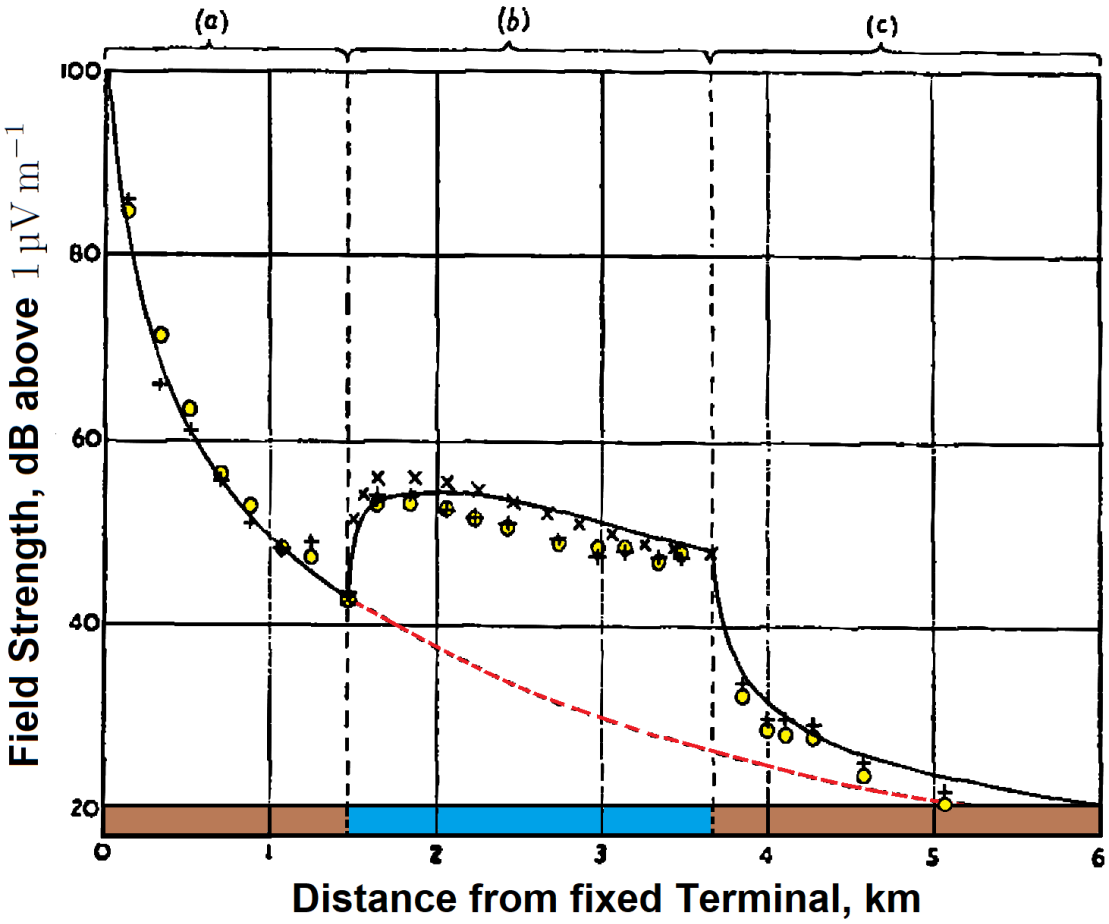

Figure 3.1.: Calculated curve and experimental results of the 4 m experiment.

—— Theoretical curve: land and sea
- - - Theoretical curve: land only
+ + Measurement at fixed terminal
○ ○ Measurements at mobile terminal
× × × Values taken from continuous record
$\lambda = 3.9\,\mathrm{m}$ (3.13 MHz)
$P = 10\,\mathrm{W}$
Vertical polarization.
$h_T = h_R = 0$
(a) Land, $\kappa = 15$ e.s.u., $\sigma = 1 \times 10^{-13}$ e.m.u.
(b) Sea, $\kappa = 80$ e.s.u., $\sigma = 4 \times 10^{-11}$ e.m.u.
(c) Land, $\kappa = 15$ e.s.u., $\sigma = 1 \times 10^{-13}$ e.m.u.

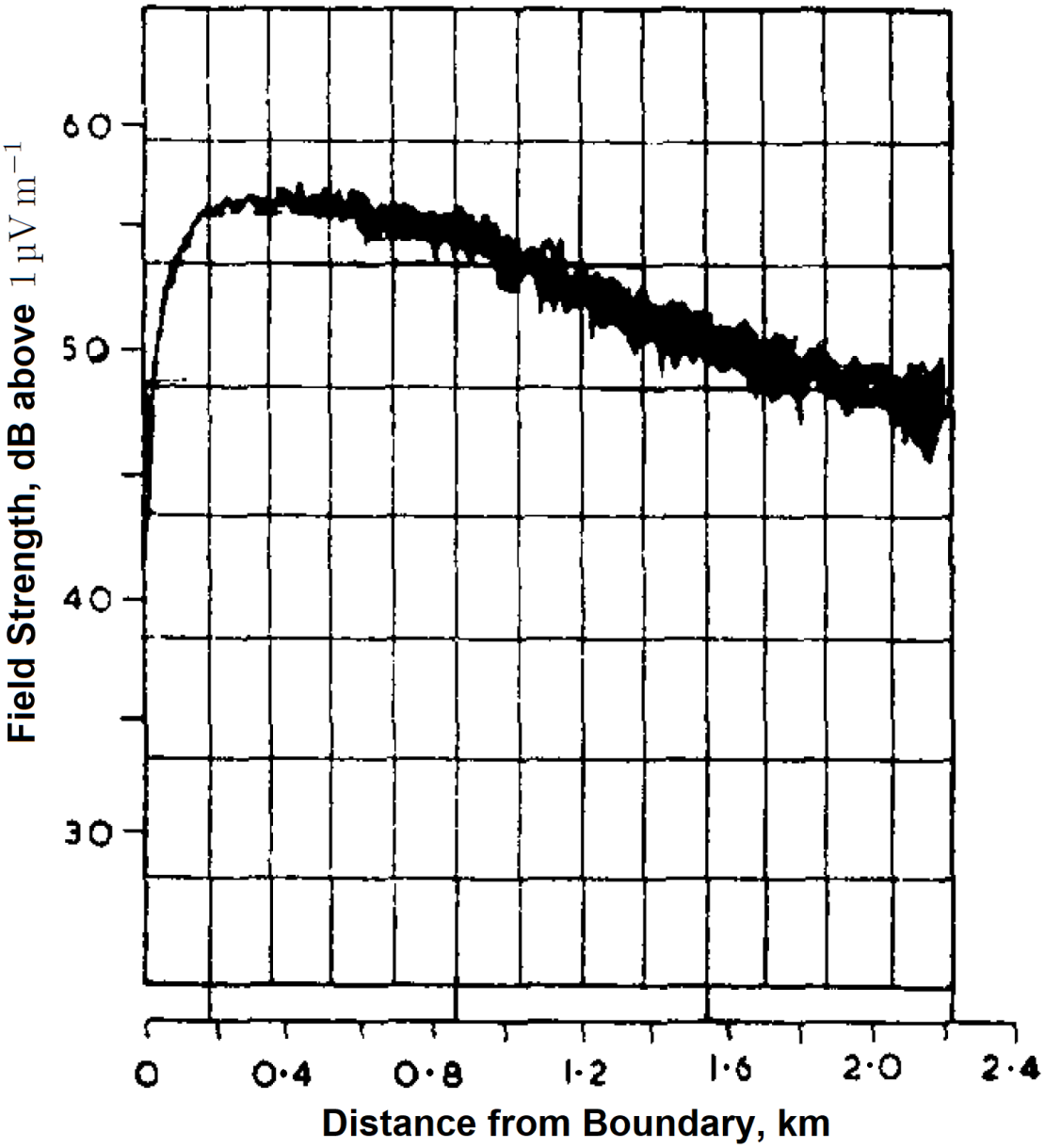

Figure 3.2.: Continuous record made at fixed terminal in the 4 m experiment.

4. The 4 m Experiment

4.1. The Site

The land and sea route chosen for these experiments, shown in Figure 4.1, lay across the River Blackwater between Northey and Osea Islands and across the marshes flanking each side of this river.
The 1.5 km of land between the fixed terminal and the river was substantially flat and free from obstructions, although a sea wall midway along the route and running parallel to it for about 180 m will be shown to have caused some disturbance.
The Blackwater is a shallow tidal river 2.2 km wide at the point selected for the experiment. The discontinuity of level between the land and water at high tide was of the order of 3 m, which was not considered to be serious enough to affect the major point at issue.
The land on the further side of the river was far from ideal because of local obstructions in the form of beach bungalows, telegraph lines and wire fences. As the checking of the rapid drop in field-strength on crossing the sea/land boundary was reckoned to be of secondary importance compared with the verification of the recovery effect, these features were not considered to be intolerable.

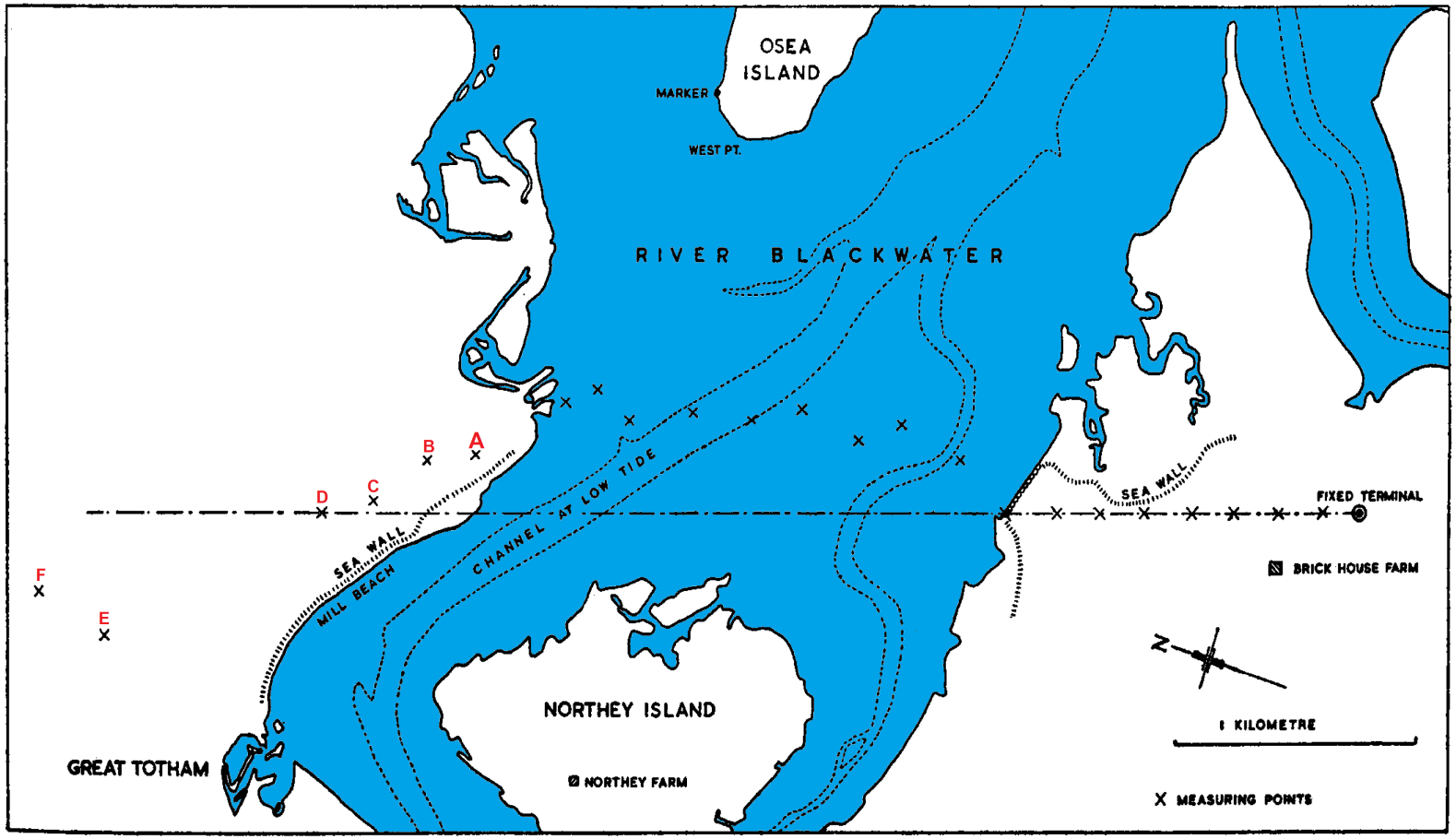

Figure 4.1.: The site of the 4 m experiment across the Blackwater. **A** to **F** = route with car,
------- = route with motor-boat

4.2. The Equipment Used

Two transmitting and receiving sets, each crystal-controlled, operating on a frequency of 77.575 MHz ($\lambda \simeq 3.9$ m) formed the basis of the experimental equipment. One set was used as a fixed terminal at the beginning of the initial land path, and the other as a mobile one to be moved away from it along the chosen route. Thus, it was possible to communicate in both directions and hence check the reciprocity condition.
The transmitter and receiver at each terminal were arranged to match into a 75 Ω coaxial feeder terminating in a vertical dipole, the lower limb of which was about 0.5 m above the ground.
The signal level was read from the anode feed of one of the receiver valves controlled by the automatic gain control. Care was taken to see that over the contemplated range of signal variation no saturation of the receiver occurred.
In addition to the transmitters and receivers, a calibrated signal generator was used to compare the receiver input-signal levels.

4.3. Method of Measuring the Field-Strength

Although the experiment did not demand an absolute measurement of field-strength, it was considered desirable. From the known aerial and feeder characteristics the field-strength could be derived from the measured voltage at the input terminals of the receivers.
Field-strength measurements were made at the fixed terminal by noting the value of the anode current modified by the presence of signal, and then immediately substituting the aerial and feeder by the 75 Ω output of the signal generator, adjusting the generator attenuator to give the same anode current as the signal and then logging its setting in microvolts. At the mobile terminal the procedure was not so straightforward, as it was not convenient to have the extra encumbrance of a signal generator. Therefore, in making the measurements, two readings of the meter were taken, namely the anode current in the presence of the signal followed by the steady value of the current without the signal. The latter was a good enough criterion of the gain condition of the receiver.
At the end of the field tests the signal generator was used to inject into the mobile receiver to match the various signal levels that had been logged; for each comparison the accumulator volts were adjusted by a rheostat to obtain the steady anode-current reading relating the gain state of the receiver to the appropriate signal reading. As a control of transmitter power, the anode current of the final stage was logged each time a signal-strength reading was made. It was found that throughout the tests the output power from each transmitter varied by less than 0.5 dB and it was assumed that the aerial and feeder conditions themselves remained stable.
In this way the greatest difficulty in making reliable measurements under the

experimental conditions, namely the variations of transmitter power and receiver gain due to the falling voltage of the accumulators, was satisfactorily overcome.

4.4. Measurement over the Initial Land Path

The mobile equipment was installed on a farm trailer and towed away from the fixed terminal towards the sea to predetermined positions where the dipole was placed well away from the equipment. Field-strength measurements were made at both terminals. Irregularities in the field-strength curve, which were suspected to be caused by reflection from the sea wall where it ran parallel and close to the route, were investigated by displacing the aerial laterally on either side of the straight-line route. The field-strength value falling nearest to the smooth curve through the points as a whole was taken as correct.

4.5. Measurements over the Sea

The mobile equipment was installed in a 25 ft motor-boat, and the same aerial as was used on land was clamped to the stern. The lower limb of the dipole was arranged to be 0.5 m above the sea.
The boat proceeded across the water from the point where the land measurements had terminated, stopping at regular time intervals for a field-strength measurement and a compass fix to be made.
On plotting the course of the boat, it was found that it had drifted considerably during the measurements, and that the route taken was well off the desired line and to the east of it, as shown in Figure 4.1. It will be seen that the length of sea wall running parallel to the desired land route, and mentioned above, lay between the fixed terminal and the boat in its displaced position, where it may have introduced a certain amount of screening.
This test did, however, verify the reciprocity condition, and this fact was made use of on the following day to employ a different and more accurate method of measuring the field-strength over the sea path. The measuring meter at the fixed terminal was replaced by an automatic recorder, and the transmitter on the boat was arranged to radiate continuously while the boat proceeded at constant speed across the water on the course indicated in Figure 4.1.
Attempts to obtain a more detailed record of the initial rise in field-strength were limited by the minimum speed of the boat and the maximum speed of the recording paper. Before leaving the site, the automatic recorder was calibrated by means of the signal generator.

4.6. Measurements over the Further Land

For this stage of the tests the mobile equipment was installed in a private car, and the aerial used in the previous land and sea tests was once more employed.
As already explained, local obstructions made this route far from ideal, and it was therefore necessary to choose measuring points considerably off the desired route, as shown in Figure 4.1 and marked **A** to **F**.
The measurements commenced at the sea boundary one hour before high tide and were completed one hour after high tide at a point about 1.4 km inland. Throughout this part of the test, the technique of measurements was the same as described for the initial land path.

4.7. Results

The results of the experiment are summarized in Figure 3.1. The theoretical land curve has been calculated for a radiated power of 10 W from a perfect dipole, with an assumed conductivity of 1×10^{-13} e.m.u. and a permittivity of 15 e.s.u. The latter is the controlling factor, and the value was chosen to give a good fit to the measured values over the initial land path. For sea, the corresponding values were taken to be 4×10^{-11} e.m.u. and 80 e.s.u. respectively, and the curves for the sea and further land sections were calculated by the process described in detail in Part III.
In a recent paper, Bremmer, [4], has put the condition that the curvature of the earth can be neglected in the simple form that $d < 5\lambda^{\frac{1}{3}}$, where d is the distance from the transmitter in km and λ is the wavelength in metres. For $\lambda = 4$ m this gives $d < 8$ km, so that the Sommerfeld analysis could be used in computing the curves.
It will be seen that the agreement over the initial land path was very good, though there is a suggestion that the field may be appreciably modified somewhat before the boundary. This effect may, however, be due to the small imperfections of the site, in particular the above-mentioned section of sea wall and the 3 m drop from the land to the sea level at the boundary. It would need a more detailed investigation on an even better site to detect with certainty any boundary effects neglected by the approximations made in the theory in this region.
The measurements made over the sea path during the first day's tests gave points lying markedly below the computed curve, and, as suggested above, this can probably be attributed to the screening effect of the sea wall due to the boat drifting off course. This is supported by the good agreement obtained for the points taken from the continuous recording made the following day. The complete record is reproduced in Figure 3.2, which gives the whole of the rapid initial rise which occurred between the boundary and the position at which the first spot reading was made. On the original record a rapid interference effect can be distinguished immediately beyond the boundary, though here again it is difficult to say whether

this is a genuine boundary phenomenon or is a diffraction pattern due to the discontinuity in level at the boundary.
The results over the further land are naturally somewhat irregular, bearing in mind how much off course some of the measuring points had to be chosen, but they are good enough to confirm the rapid drop of field-strength on crossing a sea/land boundary. At the short-wave limit, the curve in this case should settle down to the all-over-land curve, provided the first boundary is not too near the transmitter, and this condition is almost fulfilled under the conditions described in Figure 3.1.
As a whole, the experiment gives an excellent confirmation of the reciprocity condition and of the general correctness of the MILLINGTON method as applied to such short waves. It suggests that the approximation to the rigid solution is as close as practical conditions would warrant, unless, possibly, an exceptionally good site near the boundary were available.

5. The 100 m Experiment

5.1. The Site

Fortunately, on this relatively long wave, the conditions were not as stringent as for the 4 m experiment, where the flatness of the terrain and the discontinuity at the boundary were concerned, though it was essential to choose a route over which the conductivity was known to be reasonably uniform. In the interests of economy of time and expense, it was decided to exploit the possibility of using one of the regular sea routes from England. Inspection showed that quite the most likely one was the Newhaven-Dieppe route, which is a straight run of 120 km (see Figure 5.1). Admittedly it makes an angle of only 50° with the general line of the South Coast, and initially it runs rather near to the Beachy Head promontory, but it was considered that these imperfections would cause only secondary modifications to the major phenomenon under investigation.

This choice was supported by the reasonably favourable land route implied, as, apart from possible local areas of lower conductivity, the general nature of the land along the route suggests a fairly uniform conductivity of about 1×10^{-13} e.m.u. The unevenness of the ground was relatively greater with respect to the wavelength than for the 4 m site, especially as the route passed between Box Hill and Leith Hill in the North Downs and close to Ditchling Beacon in the South Downs. Here again, however, it was felt that the conditions were good enough for the purpose in hand.

The decision to work with the aid of pulses as near to 100 m as possible, and the desire to use the Newhaven-Dieppe crossing, were both fulfilled by the existence of a suitable pulse transmitter at the Radio Research Station at Slough, which was available at the time. Slough is 87.5 km from Newhaven and lies 17.5 km to the side of the Newhaven-Dieppe line. Though this means that the land path would change as the receiver was taken across the Channel, owing to the slight difference in direction of the Slough-Newhaven and Newhaven-Dieppe lines, it was considered that the effect would not be serious. It was necessary, however, to allow for the interposition of the Beachy Head promontory that increased the land path when the receiver was at Dieppe by 10 km over the value at Newhaven.

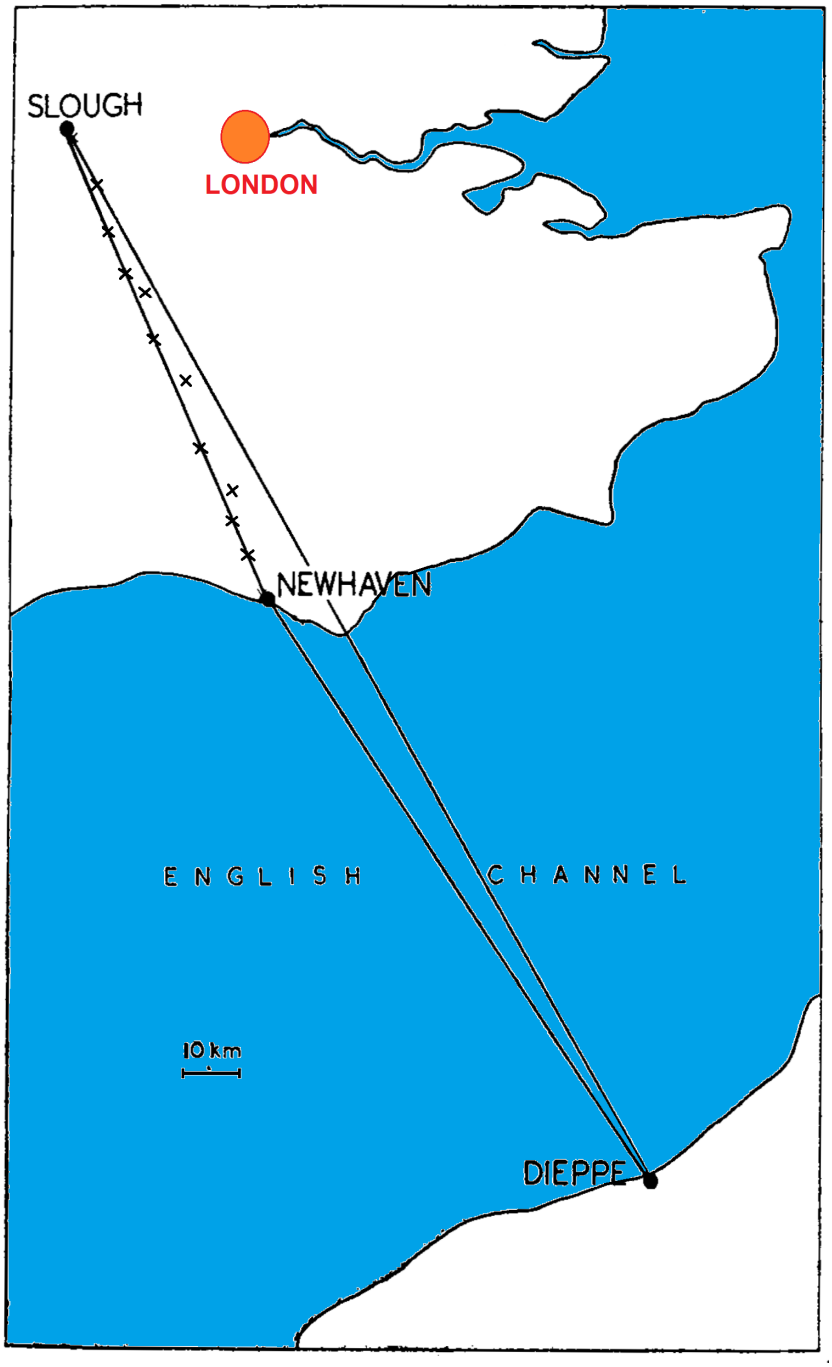

Figure 5.1.: The site of the 100 m experiment across the Channel
××× Measuring points

Figure 5.2.: RF Ammeter, exemplary illustration; MFJ-836, RF IN-LINE CURRENT/SWR/WATTMETER, 1 MHz to 30 MHz, 3 A, to determine feedpoint impedance and monitor antenna system for detrimental changes under power

5.2. The Equipment

The output of the Slough pulse transmitter was modified to work into an earthed $\frac{1}{4}\lambda$ vertical aerial, and an RF Ammeter (see as an example Figure 5.2) was connected in series to serve as an indicator of the constancy of the radiated power. The estimated peak pulse power was 10 kW. For an assumed conductivity of 1×10^{-13} e.m.u., the field-strength at Newhaven should be about 90 µV m^{-1}, which, according to the theory, was ample for the sea measurements.
In this experiment no attempt to test reciprocity was feasible, and the mobile terminal equipment consisted simply of a field-strength-measuring equipment and a miniature cathode-ray oscilloscope. The field-strength-measuring equipment employed a rotatable tuned unscreened loop aerial[1], which was calibrated by a known signal from a local oscillator. A field-strength measurement was carried out by matching the local calibrating oscillator against the peak amplitude of the received pulse displayed on the cathode-ray oscilloscope.

5.3. The Land Measurements

Sites had previously been marked off on the map every few kilometres between Newhaven and Slough, and an endeavour was made to keep to them within a kilometre or so, subject to the local conditions found on arrival.
In each locality, a search was made for the most promising site clear of overhead wires, trees and other obstructions. Once a site had been decided upon, its

[1]For loop aerial (British English) or loop antenna (American English) see e.g. `https://en.wikipedia.org/wiki/Loop_antenna` or `https://www.qsl.net/vk5br/LoopAntennas/VLF-LFLoopAerial.htm`

suitability was tested by the simple expedient of examining the ground-wave suppression by rotating the loop aerial. A true bearing suppression of 15 dB to 20 dB was considered to indicate a satisfactory site.
The first measurement was made just outside Newhaven Town on an exceptionally good site. The field-strength here was 95 $\mu V\,m^{-1}$, confirming the estimated value, and assuring that an ample signal would be available for both the land and sea measurements.
Apart from two exceptions, the measured values of field-strength between Newhaven and Slough lie within ± 3 dB of a smooth curve calculated from the theory for an assumed uniform conductivity. It was considered that the consistency of these results as a whole was satisfactory for the purpose in hand.
Throughout these tests an experienced observer was controlling the transmitter to ensure that the radiated power was constant and to see that its frequency was kept accurately to the selected value of 3.13 MHz ($\lambda = 96$ m). He similarly kept watch during the subsequent sea measurements, so that continuity of conditions was maintained throughout the experiment, even though an interval of some days was required to make the necessary arrangements after the success of the initial land measurements had been assured.

5.4. The Sea Measurements

For the sea measurements, permission was obtained to install the field-strength-measuring equipment in the wheelhouse of the Paquebot "Londres". A vertical aerial from the masthead was fed into the measuring set via a screened tuned circuit and coupling coil plugged into the socket in place of the loop. It was not practicable to calibrate the vertical aerial to enable measurements of absolute field-strength to be made; therefore all measurements on the ship were confined to relative signal strengths in the knowledge that the known field-strength at Newhaven could be used as a reasonable reference.
On the outward run measurements were made every five minutes, but the high speed of the ship, 26 kn, was not at first fully appreciated, and consequently the recovery in field-strength was well advanced before many readings had been taken. The field-strength/distance curve obtained on this run therefore lacked detail in this important region. In addition, the total gain of the equipment varied unexpectedly owing to accumulator trouble, so that the calibration was somewhat uncertain, although the general nature of the curve was not in doubt.
The distance of the ship from Newhaven along the route at any given time was obtained from a plot of her position made every 15 min by the ship's navigator, as was also done on the return journey. This method was amply accurate with the added precaution of noting the time at which the ship reduced speed on approaching Newhaven and that at which she entered the harbour.
In view of the experience gained on the outward run, it was decided to make a

compromise on the return run between conserving the accumulator power supply and obtaining the maximum amount of field-strength data. Field-strength measurements were made every 15 min for two hours after leaving Dieppe, and then every 5 min during the next three-quarters of an hour, the equipment being switched off between times. Thereafter it was left on and as many readings as possible were made during the remaining 20 min or so of the run. In this way the readings were concentrated in the region of the maximum of the recovery effect and the rapid variation associated with it. It is considered that the measurements made on the return run are completely reliable.

5.5. Results

The results of the complete experiment are shown in Figure 5.3. The theoretical land curve has been calculated for a radiated power of 10 kW and an assumed conductivity of 1×10^{-13} e.m.u. This curve fits the measured values well in view of the unevenness of the ground, with the exception of the two points already mentioned where the field-strength is unexpectedly low. A land conductivity of 1×10^{-13} e.m.u. was therefore taken in computing the curve over the sea path by the method described in Part III.

Allowance has been made for the fact that, as described, the land path changes, and increases in all by 10 km at Dieppe. This has the effect of reducing the amount of the recovery by 2 dB, but even so the maximum is 8 dB above the value at the boundary, and the field-strength at Dieppe, nearly 100 km beyond the maximum, is still somewhat greater than at Newhaven. It will be seen that the measured values lie about this curve and leave no doubt about the reality of the recovery effect and its magnitude.

There is a suggestion that the initial rise in field-strength is somewhat more rapid than that given by the predicted curve, but it would need considerably more experimental work to substantiate such a difference in the presence of the disturbances caused by the imperfections of the site. It was found also on entering the harbour at the end of the return run that there was a series of interference maxima and minima, but, as in the 4 metre experiment, it is difficult to attribute this effect to the boundary itself, in view of the many local obstacles in the harbour and the possibility of reflections from the cliffs on the Beachy Head promontory.

The results as a whole show that here also the Millington method gives a solution of the problem that is probably as close as the conditions of the experiment warrant. This confirmation of the method on a wavelength where the attenuation with distance in the diffraction region is twice as great for land as for sea strongly supports its extension to still longer wavelengths. It would have been interesting to make measurements beyond Dieppe to confirm the rapid drop in crossing the sea/land boundary, but as this follows by the reciprocity argument it was decided not to involve the extra time and expense that would have been required.

This feature has already been demonstrated in the 4 metre experiment, and there are good examples of it on longer waves in the experimental results of the Danish Posts and Telegraphs quoted by Kirke. The latter include measurements on a frequency of 540 kHz ($\lambda = 556$ m) and can be cited in support of the MILLINGTON method at longer wavelengths. It would in any case be very difficult to obtain sufficiently good conditions to carry out a well-controlled experiment on, say, 1000 m with the large distances of land and sea that would be needed, even assuming that any trouble due to ionospheric reflections could be overcome.

It is felt, therefore, that the existing evidence, supported by specific experiments described above, justifies the use of the MILLINGTON method under all practical conditions. Though it had been known for a long time that recovery effects can occur in field-strength surveys, it is fair to say that the full extent of phenomenon at a land/sea boundary had not been realized or its importance appreciated.

In the light of the experimental evidence obtained, it is useful in conclusion to point out some of the practical implications of the theory as they affect the use of ground-wave propagation for radio communication and navigation.

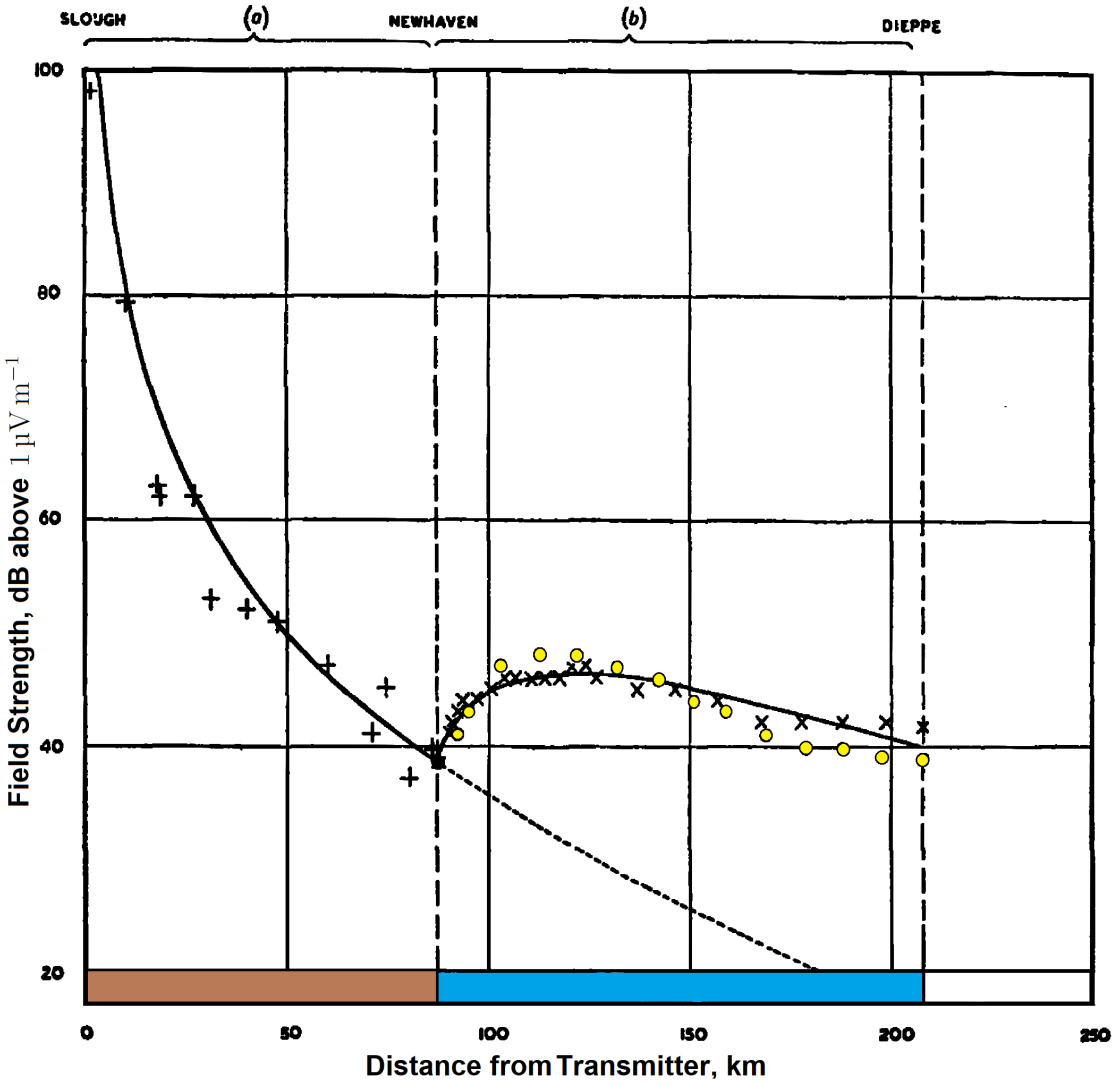

Figure 5.3.: Calculated curve and experimental results of the 100 m experiment.

—— Theoretical curve: land and sea
- - - Theoretical curve: land only
+ + Land measurements
○ ○ Sea measurements, outward run
× × Sea measurements, inward run
$\lambda = 96\,\mathrm{m}$ (3.13 MHz)
$P = 10\,\mathrm{kW}$
Vertical polarization.
(a) Land, $\kappa = 5$ e.s.u., $\sigma = 1 \times 10^{-13}$ e.m.u.
(b) Sea, $\kappa = 80$ e.s.u., $\sigma = 4 \times 10^{-11}$ e.m.u.

6. Practical Implications

6.1. Siting of Transmitters in Relation to Stretches of Water

It was pointed out in Part III that one of the consequences of recovery effect is that it should be possible under some conditions for a transmitter radiating initially over sea to produce a stronger signal over the succeeding land than if it were placed on the land itself. This was in fact demonstrated in the 4 m experiment in the making of the continuous recording, where the field-strength at the fixed point inland rapidly increased as the transmitter was moved away across the water. This phenomenon immediately implies that the service area of a transmitter radiating inland from a site near to a stretch of water may be greatly increased by moving the transmitter away across the water.

As an example, the effect is shown in Figure 6.1 of moving a 4 m transmitter, with a vertical omnidirectional aerial near to the ground, from a point T_1 on the bank of a river 0.6 km wide to a point T_2 on the opposite bank. The optimum width is determined by the distance of the maximum of the recovery effect beyond the boundary. It will be seen that the increase in the service area is quite remarkable, and these conditions may well apply, for instance, in the case of a transmitter at the central control of a police mobile service installation.

In constructing the figure, each radial was treated on its own merits, so that the true nature of the rapid field-strength changes near the boundary is somewhat in doubt, though Figure 7.2 shows that very rapid changes occur even on medium waves. This latter figure, incidentally, shows the advantage on medium waves of serving a land area from a transmitter situated on the further side of a sea channel, and this effect is also well known in connection with the reception of the B.B.C. programmes from Start Point along the South Coast and of the Radio Normandie Station at Fécamp over Southern England.

A further interesting outcome of the recovery effect is the distortion it produces in the field-strength contours surrounding an island on which a transmitter is situated. If the transmitter were at the centre of an ideal circular island, the contours of constant field-strength over the sea would be circles, and, regarded as a relief map, the maximum of the recovery effect would form a rim of high field-strength round the island. When, however, the island is elliptical, there will exist regions of high field-strength corresponding to hills on the extensions of the minor axis and saddle points on the major axis, while at greater distances the contours become

approximately elliptical in shape, with the major and minor axes crossed with respect to those of the island.
This is illustrated in Figure 6.2, which is calculated for an island whose axes are 200 km and 100 km respectively, with $\sigma = 1 \times 10^{-13}$ e.m.u. and a wavelength of $\lambda = 150$ m. Here again the shape of the contours at the boundary is uncertain, but it is obvious that accompanying these amplitude effects there must be distortions of the surfaces of constant phase. Thus, although the method does not give any explicit information about the phase changes on crossing a boundary, it is possible to obtain a warning of the troubles that are likely to occur to a phase measuring navigational aid in such a case.

6.2. Influence on Frequency Sharing

In connection with the re-allocation of frequencies to meet the greatly increased demands upon the radio spectrum, it has been proposed to share some of the marine bands with low-power land mobile services working sufficiently far inland not to create an intolerable level of interference to ships at sea. Assuming that ground-wave conditions are envisaged, it is clearly most important, whatever criterion is adopted for the permissible level of interference, that the propagation conditions should be correctly assessed in what is essentially a land/sea problem.
Apart from the Somerville method that has been used by the B.B.C., the unmodified P. P. Eckersley method has been generally adopted. In the present application, where the recovery effect is vitally involved, it may underestimate the field-strength at a given distance out at sea by many decibels, and in view of the experimental evidence given above, it is clear that it is far too optimistic in the amount of frequency sharing it would allow. It may be pointed out also that under these conditions the Somerville method can likewise overrate the frequency-sharing possibilities. In the 100 m experiment, for instance, it would have predicted a field-strength at Dieppe 14 dB below the value actually obtained. As this is a matter on which international agreement must be obtained, it is not out of place here to urge that due account of the recovery effect should be taken in adopting any given method of computing field-strengths over composite paths in the absence of a complete and rigorous solution of the problem.

6.3. Long-Wave Ground-Wave Coverage

As a final implication of the theory put forward in Part III and supported by the experimental evidence given in the present paper, it is desirable to stress the importance of the land/sea path in long-wave propagation already touched on in Figure 8.3. The replacement under these conditions of a marked recovery by a long hold-up of the field-strength does not destroy the significance of the

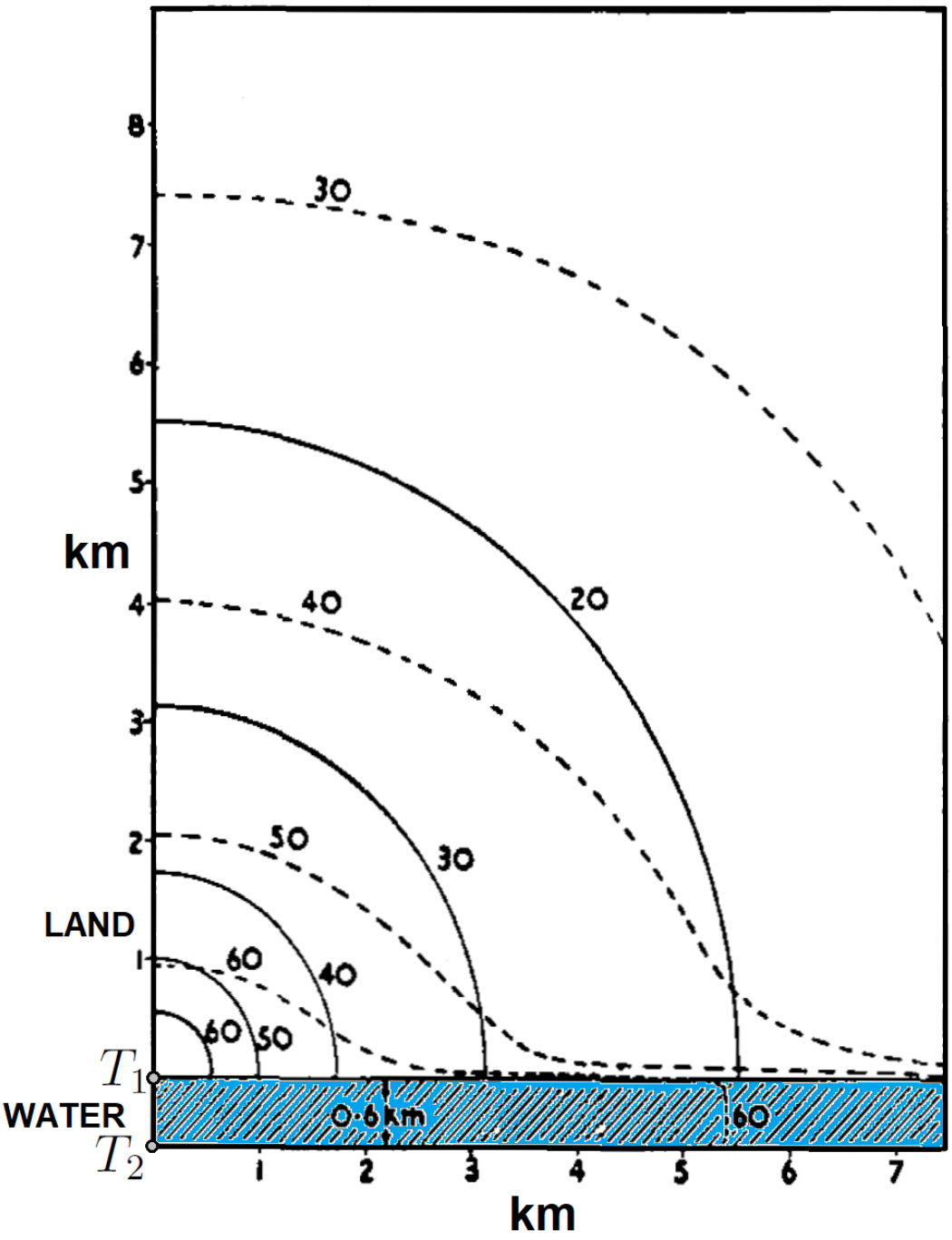

Figure 6.1.: Increase of service area on moving a transmitter away over water.
$\lambda = 4\,\mathrm{m}$
$\kappa = 15$ e.s.u.
10 W radiated.
Contours in dB relative to $1\,\mu\mathrm{V\,m^{-1}}$
—— T_x at T_1
- - - T_x at T_2

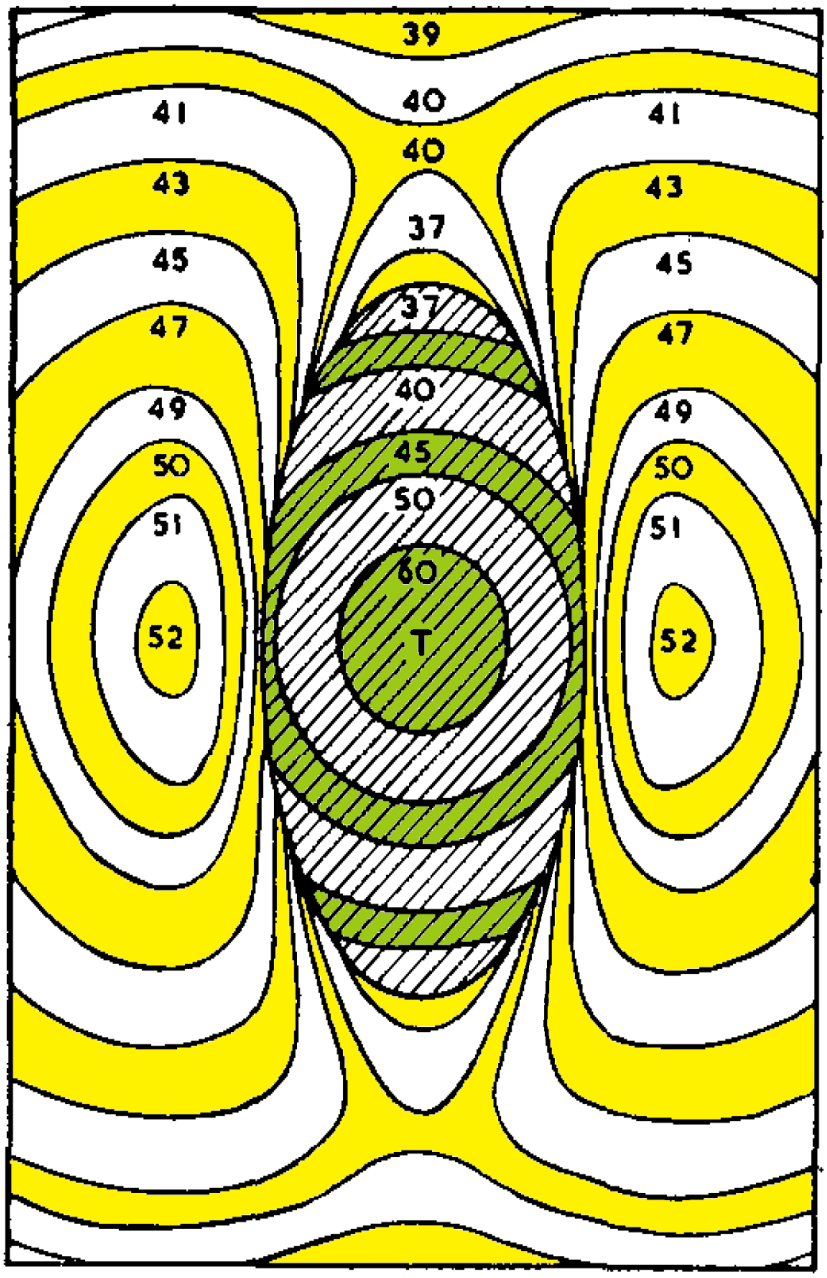

Figure 6.2.: Field-strength contours ($\lambda = 150\,\mathrm{m}$) surrounding an elliptical island with a transmitter at its centre.
dB above $1\,\mu\mathrm{V\,m^{-1}}$
$\lambda = 150\,\mathrm{m}$
$P = 1\,\mathrm{kW}$
$\sigma = 1 \times 10^{-13}$ e.m.u. for land

phenomenon, for it means that the effective ground-wave range can be increased by some hundreds of kilometres.
On these wavelengths the useful ground-wave range, even during the daytime, is governed not by the distance at which the signal becomes too weak, but by the condition that it should predominate over the sky-wave reflected from the ionosphere. If a land/sea boundary occurs where the ground-wave is just predominant for the purpose in hand, then this situation may be prolonged for a great distance out to sea. It must be remembered, however, that on still longer waves the effect disappears, as it occurs only where the overland propagation is markedly less favourable than oversea.
The use of frequencies in the neighbourhood of 300 kHz ($\lambda = 1000$ m) for navigational aids certainly shows the practical importance of these considerations, and in planning any projected scheme account should be taken of the effect of land/sea boundaries in deciding at what distance it would become unworkable because of sky-wave interference.

7. Conclusions

In conclusion it may be said that the general remarks made at the end of Part III still hold, but that the experimental evidence since obtained greatly strengthens the use of the method as a practical engineering tool for analysing transmission over composite paths. Except in the immediate neighbourhood of a boundary, where on an ideal site one might expect to be able to detect interference effects, the method gives a close approximation to the experimentally observed results so far as amplitudes are concerned, in the light of which, important practical implications necessarily follow.

8. Acknowledgments

The authors wish to thank MARCONI's Wireless Telegraph Co. Ltd., for permission to publish this paper, and they acknowledge the help of several of their colleagues in the Research Division, in particular Mr. L. J. VAN ROOYEN who was in charge of the 4 m fixed terminal and of the pulse transmitter at Slough.
They are also greatly indebted to Dr. VAN DER POL for the information contained in Figure 2.2 and Figure 2.3, and similarly to Mr. G. BRAMSLEV of the Danish Posts and Telegraphs with regard to Figure 2.1; to Dr. R. L. SMITH-ROSE of the D.S.I.R. for placing the pulse transmitter at Slough at their disposal; to Mr. F. P. BEST and Mr. C. E. TUCK of the MARCONI International Marine Co. for advice on the planning of the cross-Channel journey; to Mr. R. J. CARDY, Divisional Marine Manager, Newhaven Harbour, British Railways (Southern Region), and Commandant CHATELET, Marine Superintendent, Dieppe Harbour, French State Railways, through whose generous co-operation the arrangements for the sea passage were so speedily made; and finally to Commandant R. LETESSIER, Commandant du Paquebot "Londres" and his officers who so wholeheartedly co-operated in the sea measurements in every possible way.

9. Discussion Before the Radio Section, 15th February, 1950

Dr. R. L. Smith-Rose: This paper, which is of unusual interest to so many, presents a problem with which I have been concerned for nearly 25 years. It is a problem which dates almost from the earliest demonstrations that radio waves existed and could be transmitted great distances, over and around the earth's surface, to provide a means of communication.

Yet, in spite of the complications of ionospheric reflection, the curvature of the earth and the obstacles on its surface, the problem remains essentially that of how electromagnetic waves radiate from a vertical aerial over a flat smooth earth. I think it would not be wrong to say that we do not yet fully understand what happens within the first few wavelengths of the aerial, which, despite the increasing use of metre and centimetre wavelengths, may involve distances of 100 miles at the frequencies still in general use for broadcasting. There has, of course, been good deal of speculation on this point, which, had it provided a solution, would have eased the task of solving that problem which the authors have set. To some extent, the conditions at the boundary between surfaces of changing conductivity reproduce those near the transmitting aerial; the wavefront is set up at the boundary, and we can visualize re-radiation from it. Had this problem been solved, therefore, a solution to the authors' problem might have been reached at an earlier date.

During the past 20 years, various aspects of the problem have come into prominence in connection with direction-finding. It has long been known that, under certain conditions, waves crossing a boundary, such as a coastline, can change in direction, but I am not aware of a satisfactory explanation of that phenomenon.

It is, however, particularly from the interests of the broadcasting authorities that this problem has again been brought to our notice, and the fact that such men as P. P. Eckersley, H. L. Kirke and T. Somerville have been mentioned in the paper indicates the keen interest of the B.B.C. Since the high standards and quality of service which characterize this organization's broadcasting are such as to restrict the radius of the area served to the length of the ground wave, it is understandable why B.B.C. engineers must be so thoroughly conversant with what happens within the ground-wave conditions.

That, then, is the problem, and it is not a new one. The authors were not satisfied with the various empirical solutions which had been proposed, and, whilst their theoretical solution is not claimed to be a rigid one –by mathematical standards– it was the application of theory which enabled them to produce a theoretically sound

quasi-empirical solution from which to organize and carry out, with confidence, the experiments described in the paper. The excellent way in which these experiments were conducted can arouse nothing but admiration. Any worker who has taken radio measurements under field conditions –particularly in ships at sea– will realize the hazards which might arise and the mishaps which may occur. Yet the only unexpected occurrence was that the ship crossed the Channel more quickly than had been anticipated!

It may be said that many of their results could be deduced from certain diagrams of field-strength contours which have been published by earlier workers in this field, but, anticipating such criticism from those who would be wise after the event, the authors state that "though it had been known for a long time that recovery effects can occur in field-strength surveys, it is fair to say that the full effect of the phenomenon at a land/sea boundary had not been realized or its importance appreciated". That sentence, taken in conjunction with the experimental measurements which show a 10 dB change in field-strength, provides a very valuable appreciation of the whole situation.

In the description of the experiments carried out at a wavelength of 4 m, it is explained that the high-water level was some 3 m below the land. Since these experiments were conducted during the hour preceding, and the hour following high water, the difference in height between the land and the sea might be assumed as being nearly one wavelength, which seems to call for a correction to the calculated results in regard to height gain. Has that been allowed for in the theoretical curve?

A further point concerns the physical explanation of the occurrences at the boundary. It is known that when waves traverse a horizontal stretch of land, the wavefront is tilted forward by a measurable amount but becomes vertical at about 5λ above the earth's surface. Nevertheless, after crossing the land/sea boundary and proceeding some distance out to sea, the entire wavefront is vertical. I do not understand what happens at the boundary. How does this wavefront set itself vertically, and whence arises the resulting increase in field-strength? The authors speak of this energy as being fed downwards from the land side to the sea side, but, from the aspect of this tilted wavefront, it appears that the energy need not come from above but could arise by virtue of deflection from below.

Mr. H. L. Kirke: The effect of land-sea paths became a problem in the very early days of planning P. P. Eckersley's regional scheme of broadcasting, and first came into prominence in connection with the Washford Cross station. The next attempt to investigate the mixed path effect was made at the Burghead station in the North of Scotland. An experiment was conducted across the Moray Firth and thence via Helmsdale to Melvick. The results have not been published, as the measurements were not sufficiently accurate because of the mountainous nature of the country and the variability of conductivity. However, we did get a rise of field-strength on crossing from land to sea in the reverse direction from a transmitter at Melvick.

Proctor Wilson suggested that in all these cases there should be reciprocity, and an experiment was made in which it was found that if the whole circuit was considered as a 4-terminal network there was reciprocity.
During the early part of the last war, when certain transmissions on medium waves were intended for use overseas, there was some disagreement as to the field-strength at a distance over a land-sea-land path. The dispute was whether to use the Eckersley or the Somerville method; the former gave results which were considered to be too optimistic. It was decided to carry out further investigations, the results of which have been published, [22]. These showed that the error with the Somerville method was considerably less than that obtained when using the Eckersley method, but, as the authors point out, the method has no real justification.
I am glad that the authors have found it possible not only to treat this matter theoretically but to carry out these two remarkable and convincing experiments.
I should like to ask the authors whether their method can usefully be applied to very high frequencies radiated over uneven country, which is again a diffraction problem. The problem of calculating field-strength over such paths for the estimation of service areas of the proposed Very High Frequency (VHF) transmitters has not been solved, and empirical methods must be used, with, in many cases, surprisingly good results, although never with complete certainty.
The shadow effects of towns is mentioned in the paper. Similar effects have been noticed over London, particularly on 261 m and 203 m. In both cases there is a recovery effect on the far side of London, and the field-strength increases.
Height gain has been dealt with in the authors' experiment, and shown to be less after the recovery had taken place than before the boundary. This might well be deduced, because, after the recovery, the field-strength is higher, and, as the field-strength at a sufficient height is unaffected by the ground, the height gain in the one case must be less than in the other.

Mr. R. A. Rowden: The calculation of field-strength over mixed land/sea paths by the authors' method shows very good agreement with measured results obtained by the B.B.C. on broadcast wavelengths. The Somerville method was used by us for many years, and, when restricted to frequencies between 150 kHz to 1500 kHz and to cases where the lengths of land and sea path are not very great, e.g. within the service area of a broadcasting station, it gives results of practical value and has the advantage that it can be used with greater speed than the Millington method. On shorter wavelengths and over longer distances, the Somerville method is inaccurate, as shown by the authors' example (see section 6.2). On 300 m, however, over the Slough-Dieppe path, the Somerville method would give a result only 2 dB below the Millington value.
The authors used wavelengths of 4 and 100 m to demonstrate the large magnitude of the recovery effect experienced when passing from a land to a sea path. These wavelengths are not very much used for practical communication by the surface

wave, and I feel that the authors have tended to over-emphasize the practical importance of the recovery effect by the choice of such short wavelengths.

Mr. H. Stanesby: Three very interesting effects are thrown into relief by the authors' work, and using the methods that have been developed, the magnitude of these effects can for the first time be predicted. These effects, all three of which are well exhibited in Figure 6.3, are: (a) the recovery of field-strength found when, in travelling away from the transmitter, one passes from land to sea, (b) the bad effect of a short land path at the end of an oversea path, and (c) the good effect of a short sea path at the end of an overland path. A very early example of the recovery effect was described by Bown and Gillett in 1924, [3]. In plotting the field-strength contours of a medium-wave broadcasting station located at the southern end of Manhattan, they found that the field-strength fell to about $1\,\mathrm{mV\,m^{-1}}$ some five miles north of the transmitter, but rose again to $2.5\,\mathrm{mV\,m^{-1}}$ five miles further on in the same direction. Effect (b) has been well known for many years and is the reason why Loran (Long Range Navigation, see `https://en.wikipedia.org/wiki/LORAN`) stations, and coastal radio stations operating maritime mobile services on medium wavelengths, are placed as near to the edge of the sea as possible. In 1932, Heising, [19], showed that the effect on 5 MHz vertically-polarized ground-wave signals of moving a coastal station one mile away from the shore was the same as that caused by reducing the transmitter power in one case to 1/6 and in another case to 1/16 of the original value. It seems to be well established that, in passing from sea to land, the field-strength falls at least as abruptly as indicated in Figure 6.3. Effect (c), the beneficial effect of water or high-conductivity earth at the end of a path of lower conductivity, has also been appreciated, but not, perhaps, as fully as it deserves. It emphasizes the advantage, from the aspect of propagation, of siting medium-wave transmitting and receiving stations on mud-banks and salt-marshes, but the civil engineering problems and other practical difficulties cannot be easily overcome.
It would be interesting to know the authors' views on the probable effect of the ground constants near the transmitter and the receiver on sky-wave propagation. Presumably where a long hop is involved, the signals leave the transmitter and arrive at the receiver at relatively low angles, and it seems that the properties of the ground near the terminal points must affect substantially the overall attenuation. In view of the notable success of the ground-wave treatment, perhaps the authors could, in due course, be persuaded to consider quantitatively the sky-wave case.

Mr. P. G. Redgment: The study of ground-wave propagation is of considerable interest in connection with radio navigational aids; the theory described in this paper has an obvious application to the estimation of field-strength, and Loran –which operates on the ground wave at fairly high frequencies– is a typical case in which it should prove useful.
Some navigational aids operate by comparing phases of ground waves, and it is

important to know what disturbance of phase takes place when the wave crosses a land/sea boundary. Experience indicates that there is a disturbance, but the results are too complicated to permit analysis of the effects on a given path. The authors' results, showing a large amplitude variation, seem to suggest considerable phase variation also. In the absence of a full theoretical solution it would be of great interest –and of very considerable practical importance– were we able to get a similar semi-empirical method for estimating the approximate magnitude of the phase change.

I think that this might be approached by an extension of the authors' work on attenuation. Unfortunately, we are not concerned with a minimum-phase-shift network, and the usual relationship between phase and amplitude cannot be applied directly. However, it seems that the disturbance of amplitude and phase at the boundary could, perhaps, be regarded as perturbations due to a fictitious minimum-phase-shift network superimposed on the distributed network corresponding to normal propagation, and thus the phase shift could be calculated from the results given in the paper. I suggest that this may be well worth an investigation.

It would be desirable to check any results thus obtained, and I should like to suggest two methods for consideration. For the case of a simple circular island with a transmitter in its centre, it might be possible to construct a network analogue of the field. Suitable inductors and capacitors could form a space network, with resistors or short-circuits to represent the land or sea surface. This might enable the exact form of the field round the boundary to be explored, and thus throw more light on what is actually taking place, as well as providing a check on the approximate theory.

Another possible approach lies in the use of a model such as the authors have shown. The difficulty here is instrumental –in the measurement of phase– but a rough idea of what is happening could be obtained by providing a standard of phase by a signal fed into the detector simultaneously with the radiated signal. A standing-wave pattern, appearing as an amplitude variation, will then be found; plotting this should enable the perturbations of phase at the boundary to be ascertained.

In describing the experiments, the authors mention the difficulty of field-strength measurement on shipboard. I am surprised that the loop was less satisfactory than the open wire, because in my experience, a loop is less seriously affected than an open aerial by the various wires stays and other conducting objects found on board of a ship. I think that a more satisfactory calibration could have been made for a loop, though its effective height is changed by the presence of the ship's hull and must, of course, be determined empirically.

Mr. P. C. Clemmow: I am particularly interested in this work because recently I have been considering the problem from an analytical aspect. The case of a single boundary in a flat earth can be treated analytically, and numerical results obtained, although the process is lengthy. In this case, it appears that the authors' field-strength curve is very nearly correct; at the worst, when one of

the media is a perfect conductor and the other a poor one, it gives field-strengths which are only 2 dB too great. If this agreement seems to be fortuitous, it does give excellent theoretical backing in addition to the good experimental justification already obtained.
It is implied in the paper that the 4 m experiment comes into the "short-wave-limit" category. But is this really so? The short-wave-limit case is discussed in chapter 4. This Section demonstrates the geometric-mean formula given by Equation 4.5. For a flat earth, the proof rests essentially on the fact that ray theory is applicable for the transmitter and receiver above a certain height, using an effective reflection coefficient of -1 which is independent of the nature of the ground. A necessary criterion for the applicability of ray theory is that the numerical distance must be large. In the 4 m experiment this condition is not satisfied. Over the distances concerned, the numerical distance for the sea is only of the order unity. It is worth mentioning that the geometric-mean formula is equally applicable to phase.
Referring again to Part III of the paper (*loc. cit.*), the argument for extending the graphical process beyond the short-wave limit seems to be confined to the diffraction region. I should like to know if the author has any ideas concerning an analogous approach for the longer wavelengths in the flat earth case –to which category the 4 m experiment belongs– in view of what I have said previously.

Mr. G. Bramslev (*Denmark: communicated*): The paper is of vital importance to the broadcast engineer wishing to determine the daylight range of medium-wave stations, for although the experimental investigations described have been carried out on much higher frequencies, it is probable that one of the greatest applications of the new method will be in connection with the practical use of the Sommerfeld curves in the medium-wave band, when the earth conductivity is changing materially with distance and direction from the transmitter.
The problem of computing ground-wave field-strengths for medium waves over composite paths has been a fairly important one for Denmark because of the geographical shape of the country, and it was, in fact, a particular case where an observed error in field-strength calculation up to about 14 dB gave a false picture of the field-strength distribution, that led us to examine the problem experimentally. Since the publication of Part III of the paper, we have compared many of the experimentally determined curves with the Millington method, and have obtained such good agreement that we now employ it with full confidence over all mixed routes, even for the prediction of field-strengths over large cities, although in this case the calculated values cannot be expected to give an entirely correct picture of the field distribution.
Figure 9.1 and Figure 9.2 are typical curves from our field-strength surveys over Denmark and Sweden. In Figure 9.1, where the operating frequency was 1060 kHz, the paths consisted of an initial stretch of 60 km of sea –including two small islands of minor importance– followed by 110 km of land and a further 40 km of sea –including one small peninsula. A very rapid drop in field-strength is observed over

one part of the land, followed by a recovery of about 6 dB over the last part of this section. This recovery is continued out to sea and reaches a total of 8 dB at the final point of the observed route. In this case, we have not attempted to find agreement with the MILLINGTON method, as no means of determining the land conductivities was available. The curve is of interest, because it shows that an exceptionally rapid drop followed by a recovery can exist on medium waves, not only as caused by the high attenuation offered to radio waves by large cities, but also where the bad conductivity can be attributed to moors, etc.

Another stretch, from Kalundborg across Seeland and Sweden to the island of Bornholm, was examined more carefully on 540 kHz and 1060 kHz. It is an interesting example of the use of MILLINGTON's method along a route consisting of three land paths and two sea paths. The total length is 250 km, with the land paths of 76, 40 and 16 km, respectively, and the sea paths of 44 and 74 km. Careful measurements of field-strength on both frequencies were made at a total of 30 points, including those near the transmitter used for determining the correct value of the radiated power. The curve for the first land path showed this to have a conductivity of 1×10^{-13} e.m.u. For the next land stretch, the radiation under test could not be used for the calculation of earth conductivity, but the Swedish broadcasting station at Malmo covered this area and was kindly placed at our disposal by the Swedish Telegraph Administration, and the conductivity was found to be also 1×10^{-13} e.m.u. for this part of the route. For the island of Bornholm, it has only been possible to estimate the conductivity to be 2×10^{-14} e.m.u. from an examination of geological maps, but lack of precision in this value will only affect the shape of the field curve over the last 10 or 15 km.

The results of the measurements are shown in Figure 9.2, together with the theoretical curves given by MILLINGTON's method. It will be seen that the agreement is very good indeed, in fact better than that obtainable in many field-strength surveys over apparently homogeneous ground. These curves, and others showing nearly as good agreement, have been taken by the Danish Post and Telegraphs as proof of the practical application of the method on medium waves. There is, however, one point where we feel a little uncertain about employing MILLINGTON's method over mixed paths, namely the case of very narrow stretches of land or sea, where the conductivity changes considerably on both sides of a radial. As also pointed out by the authors, this case needs further investigation, both theoretical and practical.

I feel that we have now got a very practical and efficient engineering tool for use in the prediction of ground-wave field-strengths.

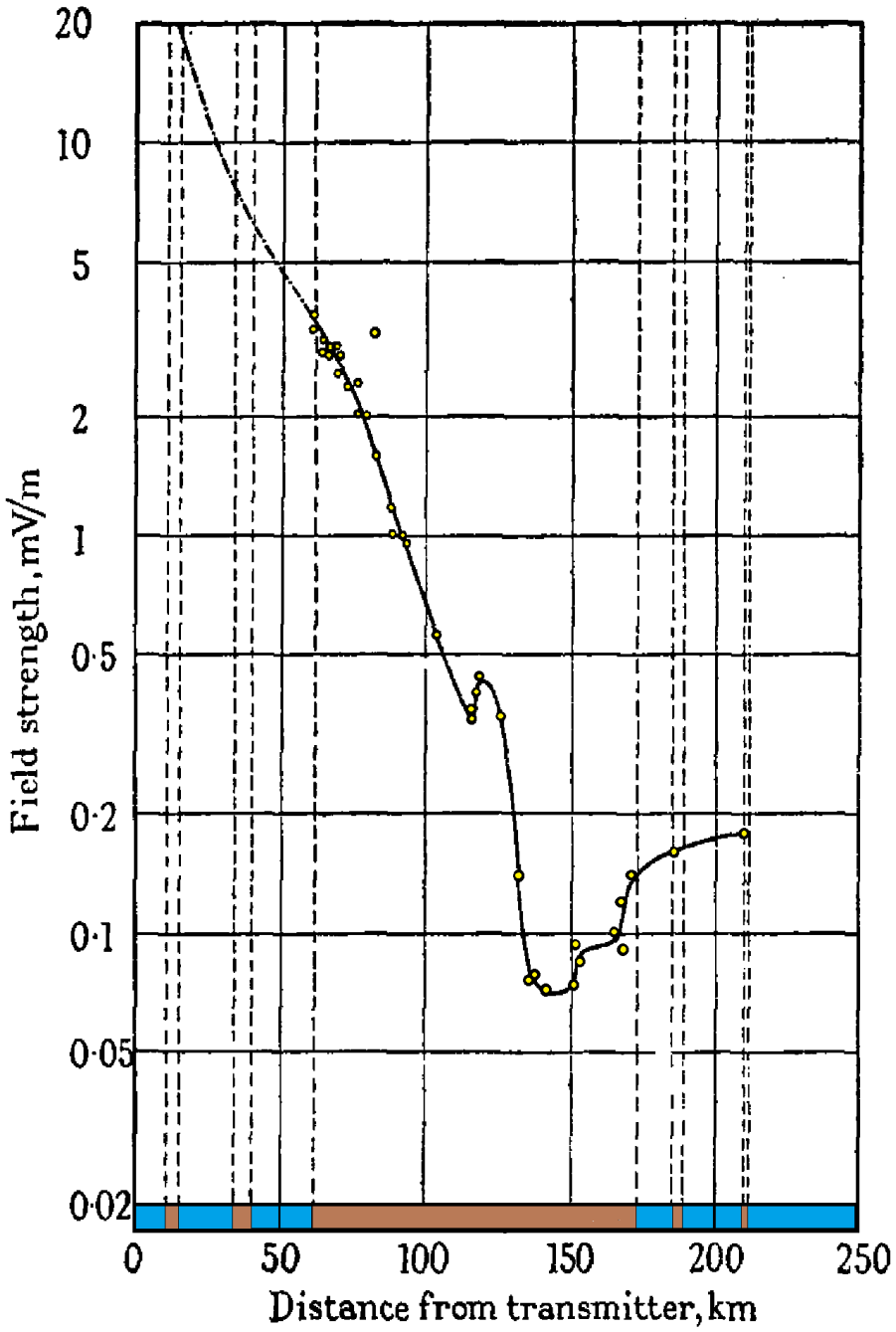

Figure 9.1.: Variation of field-strength with distance.
Route: Kalundborg-Thyborøn
Frequency: 1060 kHz
Power: 1 kW
Land paths
Sea paths
Measured values

Mr. W. Struszynski: I should like to emphasize the importance of the authors' experimental discovery of "the recovery in field-strength" effect. The results obtained provide an invaluable quantitative check on any theoretical formulæ and therefore on the validity of the assumptions and approximations involved.
Referring to the problem raised by Dr. Smith-Rose, namely that of the distortion of the wavefront, I should like to comment on the physical side of it. The wavefront assumes a certain tilt at the earth's surface, instead of being vertical. This distortion results from a constant flow of electromagnetic energy from the upper region of the wave to the lower, in order to supply energy to the wave transmitted into the earth, the flow being due to the presence of an electromagnetic field at the interface of the two media. The energy transmitted into the earth directly from the aerial is rapidly attenuated by the poorly conducting earth. Therefore the required energy can be supplied only indirectly from air close to the surface. Normals to the wavefront indicate the direction of the flow of energy in that region, and the tilt of the wavefront, therefore, shows that this flow is not horizontal at the surface (see Figure 9.3). At the earth's surface, some phase lag occurs, which results in a slight reduction of the phase velocity of the wave in comparison with that in free space. This is explained by the fact that the energy is drawn from higher regions, so that distortion of the wavefront gradually spreads from the lower region to the higher (see Figure 9.4). The phase lag at the earth's surface increases with distance, but the rate of this increase becomes smaller, and in fact the ultimate phase lag for a flat earth would tend to π (i.e. $\lambda/2$). The corresponding phase velocity would gradually increase until it approached the velocity of light for very large distances. The tilt of the wavefront changes with the height, becoming gradually vertical.
When the wavefront crosses a coast-line from land to sea, it has a large tilt corresponding to land conditions, and the downward flow of energy is still the same as it was over land. Since sea is a medium of much greater conductivity and dielectric constant than the land, less energy is transmitted into the sea, and so a considerable portion is reflected back to produce an increase of field intensity at the surface, thus explaining the effect of recovery in field intensity. At a great distance from the boundary, the conditions for transmission over the sea are established, with the corresponding tilt, since the downward flow of energy replenishes, at a sufficiently large distance, the loss in the intervening land path (see Figure 9.5). It is reasonable to expect a rapid advance of phase to accompany the field-recovery effect; this would correspond to a local increase of the phase velocity.

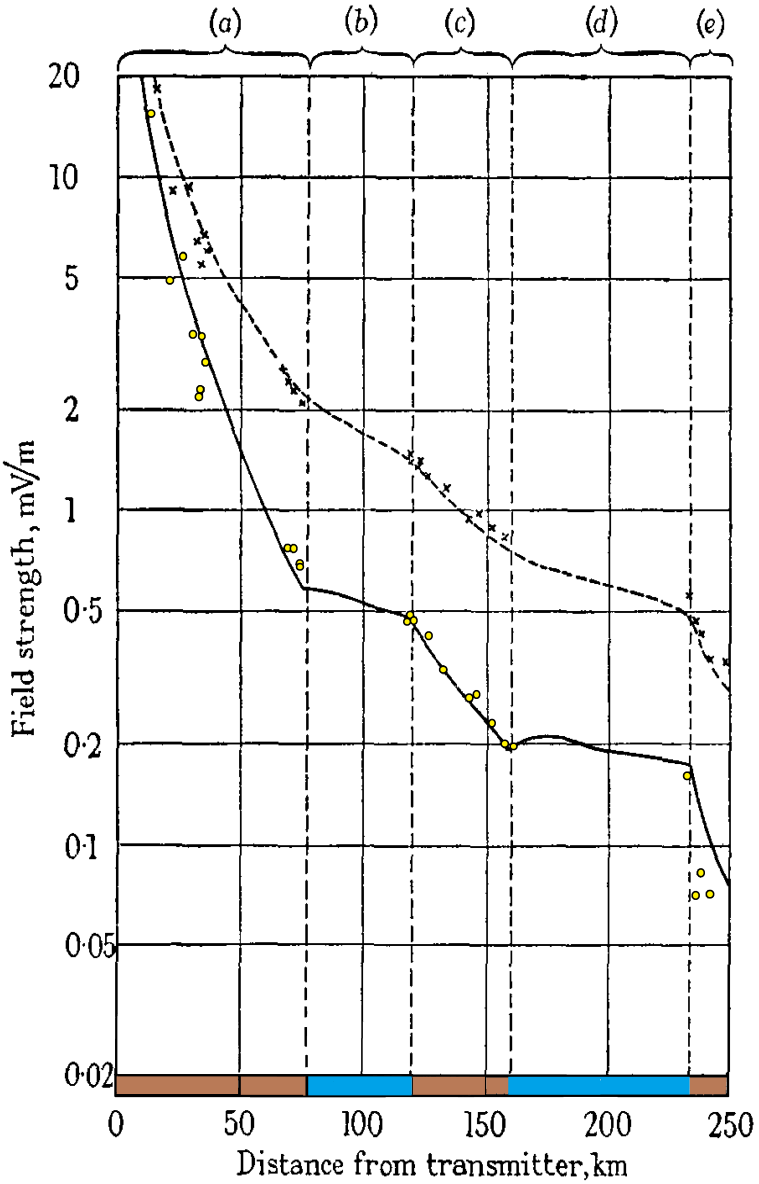

Figure 9.2.: Variation of field-strength with distance

- Land paths
- ooo Measured values, 1060 kHz
- ××× Measured values, 541 kHz
- Sea paths
- —— Curves calculated by Millington's method, 1060 kHz
- - - - Curves calculated by Millington's method, 541 kHz

(a) $\sigma = 10 \times 10^{-14}$ e.m.u.
(b) Sea path
(c) $\sigma = 10 \times 10^{-14}$ e.m.u.
(d) Sea path
(e) $\sigma = 2 \times 10^{-14}$ e.m.u.

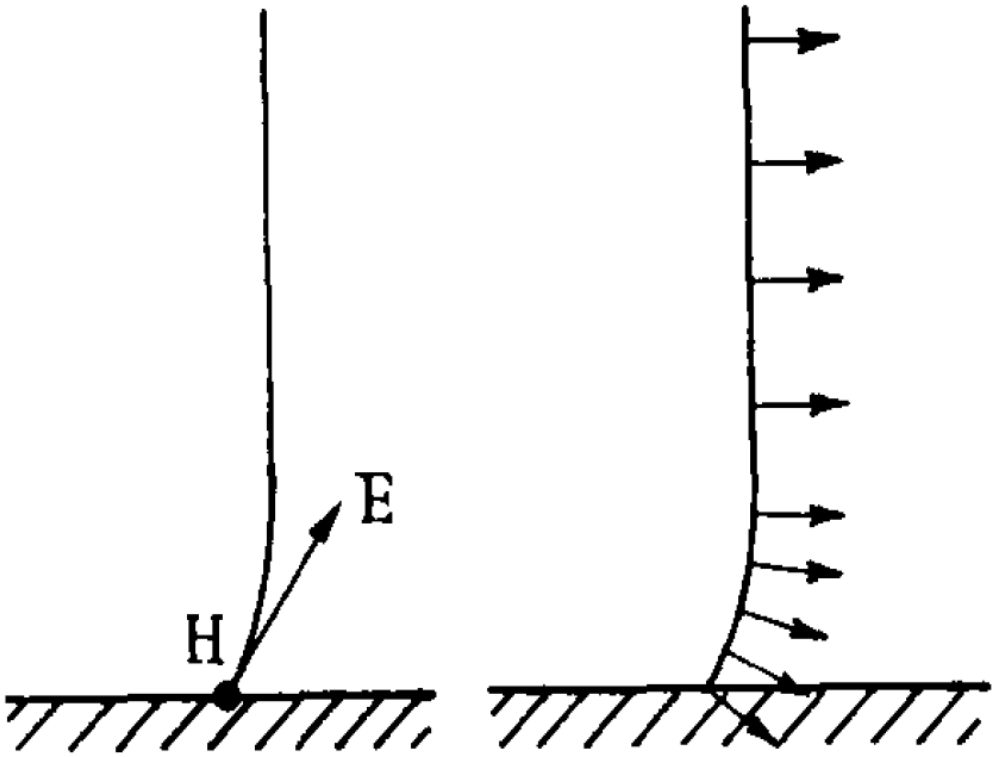

Figure 9.3.: Normals to the wavefront indicating the direction of flow of energy. Note that the direction of flow of energy is not horizontal/parallel at/to earth's surface

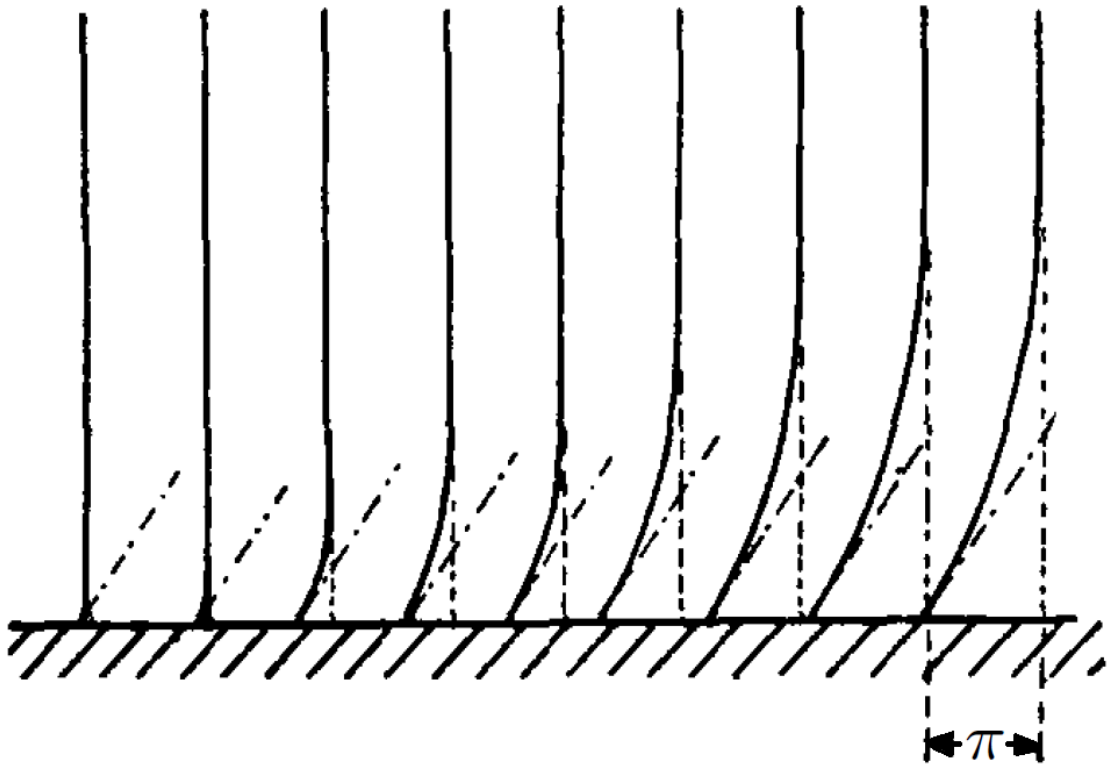

Figure 9.4.: Phase shift due to distortion of the wavefront gradually spreading from lower regions to higher ones. $\pi = \lambda/2$, which is the ultimate phase lag or shift, respectively, of a flat earth

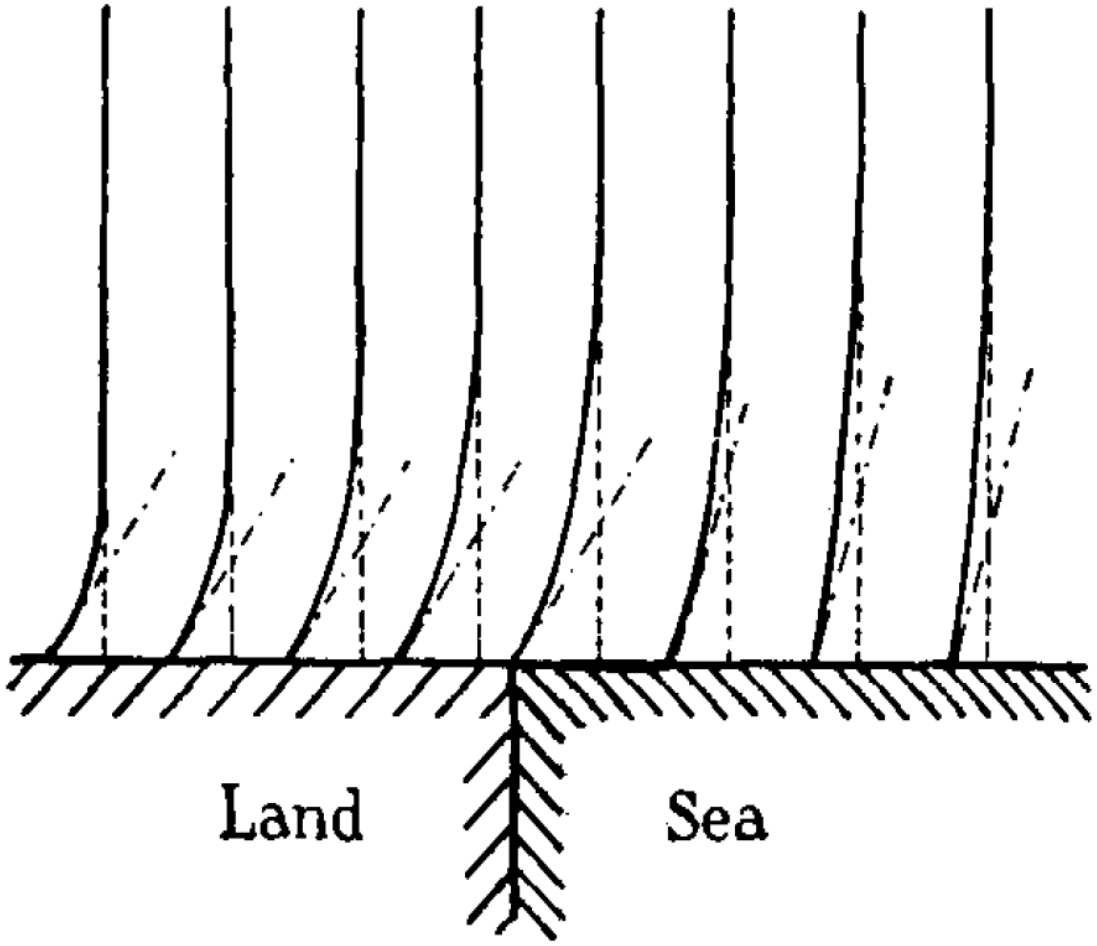

Figure 9.5.: Wavefront crossing from cost-line from land to sea

Mr. S. B. Smith (*communicated*): In Part III of the paper, Mr. Millington mentions that the method is not applicable to an approximate solution of the phase-surface conditions, which are experienced when a wave is travelling obliquely across a mixed land/sea path. Usually, an amplitude variation of this type is accompanied by a phase variation. It is realized that these boundary conditions do not lend themselves to any simple evaluation of the constant phase-surface contours. Has the author any simple method which could be applied to these phase-surface calculations and also to any phase-recovery effects which must occur at a distance from the boundary?
In Part IV of the paper, a brief reference is made to sky-/ground-ray direction finding interference errors. Some time ago I was interested in a study where a marked sea recovery was present. Under these conditions, the effects upon the night-time accuracy of a Consol[1] directional beacon were most apparent. If one considers a Consol beacon covering a uniformly conducting and smooth earth, or, for that matter, over the sea, the night-time standard deviation usually reaches a single maximum at a distance where the ground- and sky-wave field-intensities are equal in amplitude; this is due to some lateral deviations present in either or both rays. In a mixed-path case, where considerable recoveries are present, two well-defined standard-deviation maxima are to be expected, namely when the quasi-mean night field-intensities are equal to those of the ground rays at the appropriate distances. It would be interesting to know if such double-peaked standard deviation curves have been observed by those using Consol or other beacons operating along suitable paths? Have the authors any data?

Messrs. G. Millington and **G. A. Isted** (*in reply*): In answer to Dr. Smith-Rose, about the land/sea discontinuity in the 4 m experiment, it would probably be incorrect to allow for it in the height gain over the sea. It is the actual height of the receiving aerial above the sea that matters, and the difference in level at the boundary would produce a local disturbance which would modify the interference effects that may exist already. His question about the change in the tilt of the wave has been amplified by Mr. Struszynski. Undoubtedly the increase in field-strength over the sea is associated with a change in direction of the energy flow above the earth. For a complete explanation it may be necessary to study the Poynting vector, though in a composite wave system, such as exists near the surface of the earth, the vertical electric field by itself may not suffice to define the energy density.
Our work cannot help Mr. Kirke with his problem of VHF propagation over uneven country, as it is expressly limited to changes in the earth constants. Although there may be some connection between the diffraction associated with irregular

[1]Consol, meaning "by the sun" and known by its UK code name Consol, was a radio navigation system developed in Germany during World War II, see `https://en.wikipedia.org/wiki/Sonne_(navigation)` for further explanations.

terrain and that caused by inhomogeneities, we do not see how to apply our results directly to his problem. His point about the deduction of the difference in height gain on the two sides of the boundary from the recovery effect is surely only a reversal of our explanation of the latter by an appeal to the different height-gain functions over land and sea.

While we agree with Mr. ROWDEN that the SOMERVILLE method often gives a good approximation on medium waves, the remarks of Mr. BRAMSLEV show that the recovery effect can be an important feature of medium-wave broadcasting. On 300 m, the contrast between the attenuation over land and sea, which is not great for a land conductivity of $\sigma = 1 \times 10^{-13}$ e.m.u., increases rapidly when σ decreases. Thus for $\sigma = 1 \times 10^{-14}$ e.m.u. the recovery can be comparable with that obtained on 100 m with $\sigma = 1 \times 10^{-13}$ e.m.u. It may be noted that Mr. ELSON, [14], has found a marked recovery, in an aircraft flying at 1000 ft across the Norfolk coast, from a transmitter operating on a wavelength of about 300 m at Crowborough.

We realize, with Mr. BRAMSLEV that the case of transmission along a boundary needs further study, although the experimental evidence suggests that waves travelling along a river only a few wavelengths wide are markedly affected by it. In fact, within wide limits, each radial can be treated on its own merits and the field-strength contours rounded off where they cross the boundaries.

Mr. STANESBY's question about the effect of the ground constants on low-angle sky-wave propagation is difficult to answer where the methods of geometrical optics do not apply. We feel, however, that it is somewhat off the theme of our paper, as it refers to the effect of the earth constants as such and not specifically to their inhomogeneity.

With regard to Mr. REDGMENT's query about the aerial used on board ship, it must be remembered that the measurements had to be made without any preliminary trial on a ship that was on a normal run with a complement of passengers. It was found that the loop aerial would not function properly under the given conditions, but that, fortunately, the existing vertical aerial was quite satisfactory for relative measurements.

We have no data to offer Mr. SMITH in support of double-peaked standard-deviation curves resulting from the recovery effect, though it is an interesting possibility and one that may well give some indirect confirmation of our theory. On the wavelengths he has in mind, however, the phenomenon should be that of a hold-up rather than of a recovery in field-strength, and we doubt whether the standard-deviation/distance curves would show more than a single broadened peak.

Mr. SMITH and Mr. REDGMENT both raise the vital question of the phase changes across a land/sea boundary in connection with coastal refraction and phase comparison methods of navigation. It is here that the work described by Mr. CLEMMOW, [7], holds great possibilities. We are indebted to him for pointing out that where our theory gives the amplitude of the wave well beyond the boundary it also implicitly gives the phase. We regard his work as of fundamental importance, as it gives our results the rigid mathematical support they need in the immediate

neighbourhood of the boundary. It is indeed gratifying that our empirical argument in this region is so strikingly confirmed, and we can only claim that if we made a guess it was at least an intelligent one.

Admittedly our height-gain argument does not apply in the case of a perfect conductor, but the formal application of our method still leads to the correct rate of attenuation far enough away from the boundary over the conductor. It therefore seemed reasonable to apply the same empirical argument at the boundary, and to suppose that the final level of the curve is not greatly in error. We feel that it is a happy circumstance that our respective methods go so closely hand in hand. It is doubtful whether it would have been appreciated that the results of the rigid analysis can be so nearly represented by such a simple graphical process had we not arrived at the latter by an independent argument.

On the other hand, the rigid treatment is invaluable in assuring the engineer that the convenient technique provided by our method is, for all practical purposes, correct and may be applied with confidence when there is more than one boundary and where the curvature of the earth has to be taken into account. We are encouraged to hope that our method may in due course be extended to apply to the phase relationships and that it may similarly give an engineering solution that will be confirmed by the rigid theory.

Meanwhile, by again invoking the height-gain argument and considering the part of the radiated wave well above the ground, it appears that, at a sufficient distance beyond the boundary, the wave on the ground will be travelling parallel to its original direction. The distance over which a rapid phase change will take place will probably be about the same distance as the maximum of the recovery effect beyond the boundary. A map such as we have shown of the amplitude contours round an island will thus give a good indication of the region of phase disturbance.

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